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Dynamics of an elastic satellite with internal friction

Asymptotic stability vs collision or expulsion

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Asymptotic stability vs collision or expulsion

Ph.D. Thesis

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Contents

Introduction

In this thesis, we study the dynamics of an elastic body whose shape and position evolve due to the gravitational forces exerted by a point-like planet whose position is fixed in the space. The first result of the thesis is that, if any internal deformation of the body dissipates some energy, then the dynamics of the system has only three possible final behaviors:

(i) the satellite is expelled to infinity;

(ii) the satellite falls on the planet;

(iii) the satellite is captured in synchronous resonance.

By item (iii) we mean that the shape of the body reaches a final configuration, that a principal axis of inertia is directed towards the attracting planet and that the center of mass of satellite moves on a circle of constant radius.

Secondly we study the stability of the synchronous orbit. Restricting to the quadrupole approximation and assuming that the body is very rigid, we prove that such an orbit is (locally) asymptotically stable.

Some additional results on the dynamics of the body close to the synchronous orbit and some new kinematic results are also present in the thesis.

The theory of bodily tides traces its origin back to the pioneering work by Darwin [9, 10], who was actually interested in the long-time effects on the Earth's rotation of the tides generated by the Moon. He studied the following situation: consider an elastic planet, whose center of mass is fixed in space and which rotates with a fixed angular velocity. Then, put a pointlike satellite on a fixed Keplerian orbit around the center of mass of the planet. As a consequence, the planet will experience tidal distortion. If the material of the planet were perfectly inviscid, then the planet would instantaneously reach an equilibrium configuration. To account for viscosity, Darwin assumed that such a deformation has instead some delay called *phase lag*. Using also some form of the averaging principle, Darwin obtained an expression of an effective dissipation acting on the orbital and spin degrees of freedom. His argument can also be used to deduce the stability of the 1:1 resonance in the Moon-Earth system.

Darwin's work was subsequently generalized by Kaula [20] and many other authors (for instance, [1, 15, 25, 30]). Critical reviews of the work by Darwin, Kaula and followers can be found in [11–13]. In particular, Kaula has developed a theory based on the use of the Keplerian orbital elements in order to describe the tidal forces acting on the body and the corresponding reactions and phase lags. This allowed him to obtain much more effective results.

In most of the subsequent studies on the spin orbit interaction in the dynamics of a satellite, the satellite is treated as a rigid body subject to the effective dissipative force and, moreover, most of the times the orbit of the center of mass is fixed and the evolution of the spin degrees of freedom is studied. The papers [4–6] take exactly this approach. Here some KAM type results have been obtained and furthermore analytic and numerical techniques have been used in order to explain the spin orbit resonance in systems like Earth-Moon or Sun-Mercury. In particular some remarkable explanation on the 3:2 resonance observed in the Sun-Mercury system has been given.

Recently Efroimsky [12] revisited the theory of Darwin and Kaula; by using continuum mechanics he computed explicitly the expression of the phase lags to be inserted in the equation of motion.

As described above, the point of view of this thesis is much more fundamental: we assign neither the shape of the body nor its orbit, instead we consider the equations of elasticity (governing the internal dynamics of the satellite) coupled with the Newton equations governing the orbital and spin degrees of freedom and study the corresponding dynamics. We try to make as few assumptions as possible in order to understand if a general behavior appears, independently of, possibly all, the specific features of the model.

The main result is the one explained at the beginning of this abstract. The only assumption needed to get such a result pertains the nature of the dissipation acting on the internal degrees of freedom. To state it, denote by σ the stress tensor and by ε the strain tensor related to the elastic configuration of the satellite. Then, the assumption is that the constitutive relation has the form

$$\sigma = F(\varepsilon, \dot{\varepsilon}) \ .$$

In particular, we assume that the stress at a given time is only function of the strain and of its time derivative at that fixed time, and that there are no memory effects.

The proof of the main result makes use of the so-called LaSalle's principle [23], which is a generalization of Lyapunov theorem to the case where there is a nontrivial set N on which the Lie derivative of a Lyapunov function vanishes. LaSalle's principle ensures that, if the phase space is compact, then any orbit approaches the largest invariant set contained in N. For the proof we first reduce to the compact case by eliminating escaping and colliding orbits, then we show that the above invariant set is constituted by synchronous orbits.

Afterwards we study the stability of the synchronous resonance. Surprisingly enough such a study is much more complicated than the previous global one. This is essentially due to the fact that to this end one has to explicitly write the Lagrangian of the system and to study the corresponding equations. So, for this study we reduce to the quadrupole approximation, and then introduce suitable coordinates in the configuration space. In such an approximation the gravitational potential turns out to be a function of the position of the center of mass of the body, of its moments of inertia and of the orientation of the principal axes of inertia. So, it is natural to try to use such functions as coordinates in the configuration space. We prove that this is possible, but to this end we have to study separately two different situations: (1) the body has a rest configuration in which it is a sphere, (2) it has a rest configuration in which it is a triaxial body. In case (2) the analysis is simple, while in the other case the analysis is more difficult and requires the use of some properties of the spaces obtained as the quotient of a Hausdorff space with respect to the action of a finite group. The conclusion is that the wanted coordinates are a 24 fold covering of the configuration space.

Then exploiting some general principles of mechanics we write the Lagrangian of the system in the considered coordinates and prove the orbital stability of the synchronous orbit. Asymptotic stability follows by exploiting the global stability result.

It is worth mentioning that in the case of a spherical body the result does not imply that the body eventually stops deforming, but only that its shape is such that a principal axis of inertia always points towards the planet. The axes of inertia could slide indefinitely in the body.

Chapter 1 of the thesis is devoted to a brief review about the existing theories which concern bodily tides and spin-orbit resonances.

Chapter 2 and Chapter 3 contain the original contribution to knowledge that is present in this PhD. thesis. Chapter 2 is devoted to the proof of the main result of this thesis, i.e. the fact that a satellite must escape, collide or get trapped in the synchronous resonance. The content of Chapter 2 is summarized in the paper

- E. Haus. *Asymptotic behavior of an elastic satellite with friction*, preprint.

Chapter 3 contains the proof of the orbital stability of the synchronous resonance. The case of a triaxial body is studied in

- D. Bambusi and E. Haus. *Stability of the synchronous resonance for the dynamics of a viscoelastic satellite*, preprint.

The result for a body with spherical symmetry is obtained in

- D. Bambusi and E. Haus. *Asymptotic stability of spin orbit resonance for the dynamics of a viscoelastic satellite*, http://arxiv.org/abs/1012.4974, submitted.

Chapter 1

The classical theories: a short review

1.1 General facts

It has always been a well known fact that the Moon always shows the same face to the Earth. This happens because the period of rotation of the Moon about its axis equals the period of revolution of the Moon around the Earth. The fact that the two periods are *exactly* equal already suggests that it should not be a coincidence. Moreover, in the last decades more and more data have become available thanks to explorations of the Solar system, which have shown that most major satellites in the Solar system always point the same face towards their mother planet. The heuristic explanation for this phenomenon has been known for a very long time. The gravitational field generated by a celestial body is not constant in space. Since even solid bodies (like many satellites are, at least as a first approximation) are slightly deformable, the satellite experiences some deformation due to the non-constancy of the gravitational field generated by the planet. In particular, the satellite gets slightly stretched towards the planet, such an elongation making the situation where the satellite always shows the same face to the planet physically stable. Such a deformation of the satellite is originated by the same type of effect which generates ocean tides on the Earth. For this reason, the situation in which a celestial body always shows the same face to the body it orbits is often referred to as "tidal locking". In a wider context, one may be interested in the so-called spin-orbit resonances. A spin-orbit resonance occurs whenever the ratio between the period of rotation and the period of revolution of a given celestial body is a rational number. The situation of tidal locking therefore corresponds to the 1:1 spin-orbit resonance (and zero inclination of the axis of rotation), also called *synchronous resonance* for obvious reasons. In the Solar system, the main body which is in a spin-orbit resonance different from the synchronous one is Mercury, which is caught in a 3:2 resonance around the Sun, meaning that Mercury's orbital period equals three halves its rotational period.

If, on the one side, the heuristic explanation of the phenomenon of tidal locking is rather simple, on the other side the rigorous mathematical investigation is very complicated. In fact, what in principle one should do is to study a complicated system of partial differential equations describing the evolution of the internal configuration of the bodies, coupled with ordinary differential equations describing the evolution of the orbital and rotational parameters. Such an investigation is made even more complicated by the fact the inner structure of celestial bodies is often very complicated and never known with absolute precision.

Since it is too difficult to find the solutions to the complete problem of motion of a deformable body in a gravitational field, the existing theories of bodily tides always make use of some relevant approximations. In the next section, we will make a brief review of the history of the classical theory of bodily tides. The huge amount of work which has been spent on the subject makes it impossible to perform a complete review, so we will focus on the main aspects of the theory, in order to clarify the distinction between the classical approach to the theory of bodily tides and the approach that we will follow in the present PhD. thesis.

1.2 Bodily tides: the Darwin-Kaula approach

In this section we give a brief description of the classical Darwin-Kaula theory of bodily tides, which has set the basis on which many authors ([14–17, 21, 22, 25, 27–29, 32, 34]) have worked in the last decades, in order to understand the main effects of the tidal friction in the Solar System. For a much more detailed review of Darwin's theory, see [13], whose notation we will follow in this section. A critical review of the different techniques which have been developed in order to explore the consequences of torques due to bodily tides can be found in [11].

When developing his theory of tides, Darwin was actually interested in studying the long-time effects on the Earth's rotation of the tides generated by the Moon.

The situation studied by Darwin ([9,10]) is the following: consider a central, deformable body of mass m, whose center of mass is at the origin, and an outer pointlike mass M, which is responsible for the deformation of m. At first, m is considered to be a homogeneous and perfect inviscid fluid, which assumes the equilibrium configuration under its own gravity and the external gravitational forces due to M, which generate tides on m. Let $\mathbf{r} = (r, \theta, \varphi)$ (spherical coordinates) be the vector representing the position of M. If now we neglect the polar flattening due to the rotation of m and we consider only the main term of the tide generating potential (i.e., if we neglect terms beyond quadrupole in the multipole expansion of the potential generated by M), the equilibrium configuration is a Jeans spheroid (an ellipsoid whose three semi-axes satisfy $A > B = C$), whose semi-major axis A is directed towards M and whose prolateness is given by

$$\varepsilon := \frac{A}{B} - 1 = \frac{15}{4} \frac{M}{m} \left(\frac{R}{r} \right)^3 , \tag{1.2.1}$$

where R is the mean radius of m (see [33]). We note that Darwin used the Jeans spheroid as equilibrium configuration, because at that time Love's theory [24] was not available yet. Using Love's theory, it is possible to include also polar flattening in the equilibrium configuration [7], but the results that one obtains are essentially the same as for the Jeans spheroid.

Then consider an arbitrary point $\mathbf{r}^* = (r^*, \theta^*, \varphi^*)$ in space. The gravitational potential generated by the prolate spheroid of equation (1.2.1) is given by

$$U(\mathbf{r}^*) = -\frac{Gm}{r^*} - \frac{k_f GMR^5}{2r^3 r^*}(3\cos^2\Psi - 1) \ , \tag{1.2.2}$$

where Ψ is the angle between $\mathbf{r}$ and $\mathbf{r}^*$ and k_f is the parameter that in the modern language is called *fluid Love number*. In the case of a homogeneous sphere, $k_f = \frac{3}{2}$.

In Darwin's theory, the assumption is made that the body M orbits m *on a fixed Keplerian orbit* of semi-major axis a, eccentricity e and inclination I. Then, the position $\mathbf{r}$ of M is a function of the orbital elements. Therefore, one can write the expression of the potential generated by the deformed m as a function of the orbital elements of M. Define the tide raising potential as

$$U_2^0(\mathbf{r}^*) := U(\mathbf{r}^*) - \frac{Gm}{r^*} \ .$$

In terms of the mean anomaly l, which is a linear function of time

($l = nt + l_0$), one has, to second order in e and I,

$$\begin{aligned} U_2^0 \quad = \quad & -\frac{3k_f GMR^5}{4a^3 r^{*3}}\left[-\frac{2}{3} - e^2 + \left(1 + \frac{3}{2}e^2 - \frac{1}{2}S^2\right)P^2 + \right. \\ & + \left(1 - \frac{5}{2}e^2 - \frac{1}{2}S^2\right)P^2\cos(2\varphi^* - 2l - 2\omega) + \\ & + \frac{7}{2}eP^2\cos(2\varphi^* - 3l - 2\omega) + \\ & - \frac{1}{2}eP^2\cos(2\varphi^* - l - 2\omega) + \frac{17}{2}e^2P^2\cos(2\varphi^* - 4l - 2\omega) + \\ & - (2 - 3P^2)e\cos l - \left(3 - \frac{9}{2}P^2\right)e^2\cos 2l + \\ & + QS[\sin\varphi^* - \sin(\varphi^* - 2l - 2\omega)] + \\ & \left. + \frac{1}{2}P^2S^2[\cos 2\varphi^* + \cos(2l + 2\omega)] \right] \end{aligned} \tag{1.2.3}$$

where we have used the notation $S = \sin I$, $P = \sin\theta^*$, $Q = \sin 2\theta^*$ and we have denoted by ω the argument of the periapsis. Observe that such a tide raising potential is a function of time through l.

If now one is interested in the tide raising effect of the potential U_2^0 on the central body m, one has to think of the point $\mathbf{r}^*$ as co-rotating with m, i.e. such that r^* and θ^* are constant, while the longitude φ^* is given by $\varphi^* = \Omega t + \varphi_0^*$, Ω being the angular velocity of rotation of the body m.

In this way, the expression (1.2.3) becomes a function of time through both l and φ^*, where one can recognize the sum of periodic terms with nine different frequencies.

There comes the main idea in Darwin's work: the *phase lag*. Up to now, we have assumed that the deformable body were perfectly inviscid and that it instantaneously reached the equilibrium configuration. Darwin, in order to take into account the effects related to viscosity, made the following assumption: the deformable body reacts to the tidal action, but it does with some delay due to its viscosity. In particular, since the potential U_2^0 is the sum of time-periodic terms with different frequencies, a specific delay is added for each periodic term. If Φ_i is a generic time-periodic argument, then the procedure conceived by Darwin consists in replacing Φ_i with the "delayed" term $\Phi_i - \varepsilon_i$, and then considering the first order approximation in the lags in the following way:

$$\cos(\Phi_i - \varepsilon_i) = \cos\Phi_i + \varepsilon_i \sin\Phi_i \tag{1.2.4}$$

$$\sin(\Phi_i - \varepsilon_i) = \sin\Phi_i - \varepsilon_i \cos\Phi_i \ . \tag{1.2.5}$$

Then, plugging the lags into the expression of U_2^0, one finds that the quardupole term of the gravitational potential becomes

$$U_2 = U_2^0 + U_{lag} \ , \tag{1.2.6}$$

where the correction U_{lag} due to the delayed response of the deformable body is given by[1]

$$
\begin{aligned}
U_{lag} \quad = \quad & -\frac{3k_f GMR^5}{8a^3 r^{*3}} \left[P^2\varepsilon_0(2-5e^2-S^2)\sin(2\varphi^*-2l-2\omega)+\right. \\
& +eP^2(7\varepsilon_1\sin(2\varphi^*-3l-2\omega)-\varepsilon_2\sin(2\varphi^*-l-2\omega)+ \\
& +17e^2P^2\varepsilon_3\sin(2\varphi^*-4l-2\omega)+P^2S^2\varepsilon_4\sin(2\varphi^*)+ \\
& -e\varepsilon_5(4-6P^2)\sin l-3e^2\varepsilon_6(2-3P^2)\sin(2l)+ \\
& +P^2S^2\varepsilon_7\sin(2l+2\omega)+ \\
& \left.+\,2QS(\varepsilon_8\cos(\varphi^*-2l-2\omega)-\varepsilon_9\cos\varphi^*)\right] \; . \qquad (1.2.7)
\end{aligned}
$$

Then the field associated to tidal forces generated by the deformed body in any point of the space can be easily calculated as the gradient of the gravitational potential $U_2 = U_0^2 + U_{lag}$. As one expects, the computation of the gradient of U_0^2 gives a purely radial field, since in the absence of lags one would have an equilibrium configuration which does not involve any torque.

Then, evaluating the so-obtained tidal field in the point $\mathbf{r} = (r, \theta, \varphi)$ where the body M is placed, and multiplying it by its mass M, one has the tidal force $\mathbf{F}$ acting on M, generated by the tide raised on m by M itself.

Because of the presence of the lags, the tidal force $\mathbf{F}$ is not aligned

[1] When considering the effects of friction, we should also have replaced the *static* Love number k_f with its dynamical counterparts, in order to take into account some attenuation of the tidal response due to viscosity. Anyway, for simplicity, we are neglecting this aspect in this short summary of Darwin's theory.

with $\mathbf{r}$, a fact which generates a non-zero torque

$$\mathcal{M} = \mathbf{r} \times \mathbf{F} \ .$$

This is actually the machinery for obtaining expressions of the tidal forces and torques, in Darwin's theory. From these expressions, using conservation laws and averaging the torque $< \mathcal{M} >$ over one orbital period, one can get expressions for the secular variations of the orbital elements of M and of the rotation of m, and expressions for the energy dissipation.

We do not enter the details of these calculations, which are very well explained in [13]. We limit ourselves to writing down the expressions obtained through Darwin's theory. Denoting by C the moment of inertia related to the axis of rotation of m, by J the inclination of the axis of rotation of m and by E the mechanical energy of the system, the following expressions are obtained.

$$< \dot{\Omega} >= \frac{3k_f GM^2 R^5}{8Ca^6}[4\varepsilon_0+e^2(-20\varepsilon_0+49\varepsilon_1+\varepsilon_2)+2S^2(-2\varepsilon_0+\varepsilon_8+\varepsilon_9)] \tag{1.2.8}$$

$$< \dot{J} >= \frac{3k_f GM^2 R^5}{4C\Omega a^6} S(\varepsilon_0 + \varepsilon_8 - \varepsilon_9) \tag{1.2.9}$$

$$\begin{aligned} < \dot{n} > \quad = \quad & -\frac{3n}{2a} < \dot{a} >= -\frac{9n^2 k_f MR^5}{8ma^5}\left[4\varepsilon_0 - 4S^2(\varepsilon_0 - \varepsilon_8)\right] + \\ & -\frac{9n^2 k_f MR^5 e^2}{8ma^5}\left(20\varepsilon_0 - \frac{147}{2}\varepsilon_1 - \frac{1}{2}\varepsilon_2 + 3\varepsilon_5\right) \end{aligned} \tag{1.2.10}$$

$$
\begin{aligned}
< \dot{E} > \quad = \quad & \frac{3k_f GM^2R^5}{8a^6}\left\{n\left[4\varepsilon_0 - 4S^2(\varepsilon_0 - \varepsilon_8)\right] + \right. \\
& + ne^2(-20\varepsilon_0 + \frac{147}{2}\varepsilon_1 + \frac{1}{2}\varepsilon_2 - 3\varepsilon_5) + \\
& -\Omega\left[4\varepsilon_0 + e^2(-20\varepsilon_0 + 49\varepsilon_1 + \varepsilon_2) + \right. \\
& \left.\left. +\ 2S^2(-2\varepsilon_0 + \varepsilon_8 + \varepsilon_9)\right]\right\} \qquad (1.2.11)
\end{aligned}
$$

$$
< \dot{e} >= -\frac{3nek_f MR^5}{8ma^5}\left(2\varepsilon_0 - \frac{49}{2}\varepsilon_1 + \frac{1}{2}\varepsilon_2 + 3\varepsilon_5\right) \qquad (1.2.12)
$$

$$
< \dot{I} >= \frac{3nk_f SMR^5}{4ma^5}(-\varepsilon_0 + \varepsilon_8 - \varepsilon_9)\ . \qquad (1.2.13)
$$

Many decades after Darwin, Kaula [20] made a remarkable generalization of Darwin's work. Kaula computed tidal forces and torques following Darwin's approach; however, whereas Darwin stopped to quadrupole terms in the multipole expansion of the tidal potential, Kaula was able to deduce an impressive formula (see [11], equations (20) and (21)) expressing the complete multipole expansion of the tidal potential in terms of the orbital elements of M. Kaula's contribution has been so relevant that the theory which consists in introducing the phase lags in Kaula's expression of the tidal potential and deducing dynamical consequences, in a way similar to that explained above, is commonly referred to as Darwin-Kaula theory.

1.3 Physical origin of the phase lags

The classical Darwin-Kaula approach has the advantage of being very general, since no assumptions are done about the values that must be given to the phase lags ε_i. However, in order to make a rigorous physical theory of bodily tides, starting from first principles, one should do the following: (i) understand the physical origin of the phase lags, (ii) study a realistic model of deformable body and deduce the expressions for the phase lags as a function of the deformable body's rheology.

In order to understand the origin of the phase lags, it is useful to think of the analogy with a damped harmonic oscillator, with a periodic external forcing. If one has a damped oscillator with a sinusoidal forcing of the type

$$\ddot{x} + 2\zeta\omega_0\dot{x} + {\omega_0}^2 x = A\sin(\omega t) \ , \tag{1.3.1}$$

then the solution is the sum of a transient solution, which depends on the initial conditions and goes exponentially to zero, and a steady state solution, which is independent of the initial conditions. The steady state is

$$x(t) = \frac{A}{B\omega}\sin(\omega t + \phi) \ , \tag{1.3.2}$$

where

$$B = \sqrt{(2\omega_0\zeta)^2 + \frac{1}{\omega^2}({\omega_0}^2 - \omega^2)^2}$$

and

$$\phi = \arctan\left(\frac{2\omega\omega_0\zeta}{\omega^2 - {\omega_0}^2}\right) \ .$$

As we can see from these expressions, the steady state is an oscillation which has the same frequency as the external forcing, but is responding with a delay ϕ.

In this trivial example, the harmonic oscillator plays the role of the deformable body close to its equilibrium configuration, the external forcing corresponds to the disturbing potential generated by the point mass M, which in the Darwin-Kaula approach is actually a sum of infinitely many time-periodic terms (since M revolves periodically on a Keplerian orbit), and the phase shift ϕ plays the role of the phase lags of the Darwin-Kaula theory.

In [12] a very detailed explanation of the origin of phase lags is given, and, using techniques from continuum mechanics, the expressions of the phase lags for some relevant rheological models (namely the Maxwell model and the Andrade model) are obtained.

1.4 Our approach

The applications that were developed starting form the Darwin-Kaula approach turned out to be an effective tool for achieving a very good understanding of many aspects of tidal torques and tidal dissipation in the Solar system. However, from a mathematician's point of view, in such an approach there are many assumptions that need to be justified in a rigorous way. Most notably, the whole mechanism of deduction of forces and torques in then model relies on the assumption that the motion occurs on a fixed Keplerian orbit. This is certainly "almost true" for all major bodies in the Solar system, within certain time scales. However, there is no a-priori reason to expect that such an approximation is good when working on much longer time scales, for instance comparable to the lifespan of the Solar system. Even computer simulations [8] seem to suggest that on very long time scales the dynamics of the Solar system is more likely to appear irregular and chaotic rather than steady and ordered.

For this reason, in the present thesis, we deal the problem of stability of the synchronous resonance within a much more fundamental setting. Since we are interested in studying the asymptotic stability over an infinite amount of time, we need to get rid of those approximations which, despite being *very good* approximations on time scales which are not too long, are inadequate for studying the behavior of celestial bodies over infinite times. Since we approach the problem of stability of the synchronous resonance, and since typically in the Solar system such a behavior is exhibited by satellites orbiting their mother planet, we are interested in proving that the tidal deformation of the satellite itself stabilizes the synchronous resonance. Therefore, since in the planet-satellite system the satellite has a smaller mass (usually much smaller), we consider a different setting from the Darwin-Kaula one already in the fact that in our model the pointlike mass is supposed to be immobile, while the deformable body (which models the satellite) is supposed to be free to move in space. What is most important is that *we will not make any assumption about the motion of the center of mass of the satellite* (except the fact that the motion is planar, and only for the results of orbital stability of Chapter 3). In Chapter 3, we will make the planar restriction, which actually corresponds to an assumption of zero inclination of the axis of rotation of the satellite. However, we would like to point out that such an assumption has been done in this thesis with the only aim of simplifying the form of the equations of motion and consequently simplifying the study of the properties of the dynamical system. There is no obstruction in using the same formalism that we have developed in Chapter 3 for a system where the planar restriction has been removed. On the contrary, the study of such a complete 3-dimensional system is a very natural future development of the results obtained in this thesis. Another point is that, by considering the planet as a pointlike mass, we actually neglect the effects of tidal deformation and dissipation in the planet: again, there is no formal obstruction to the application of our techniques for the study of a system consisting of two deformable bodies. The introduction of a deformable planet would only result in a complication of the equations of motion.

It is worth making some more comments on the existing literature, in order to better explain why the results that we obtain are not in contrast with the already existing ones. In particular, in [4–6] a series of studies is conducted about the stability of spin-orbit resonance. The point of view that is taken here is the following: consider a rigid body, whose center of mass revolves around an immobile pointlike mass on a fixed Keplerian orbit. The rotational motion of this rigid body has zero inclination and is subject to some effective friction, which modifies the speed of rotation of the satellite and which is calculated according to some applications of the classical Darwin-Kaula theories. In such a context, the stability of spin-orbit resonances is investigated, and evidence is found that the eccentricity of the orbit is a very relevant parameter in the selection of the spin-orbit resonance in which the satellite might get trapped. For instance, on a circular orbit the satellite will fall on a 1:1 resonance, while with eccentricity $e = 0.205$ (the eccentricity of Mercury) the satellite is quite likely to fall into the 3:2 resonance, which is (locally) asymptotically stable. As we have anticipated in the introduction, the main result of this thesis, which is explained in Chapter 2, rules out the possibility that a non-synchronous resonance is asymptotically stable. Where is the contradiction? Actually, there is no contradiction. The method used in [4–6] imposes that the satellite moves on a fixed Keplerian orbit. The vanishing of the effective friction in this model corresponds to the fact that $< \dot{\Omega} >= 0$, in terms of Darwin's theory explained above. Anyway, it may very well happen that $\dot{\Omega} = 0$, but that, at the same time, secular changes in the parameters of the Keplerian orbit occur. After a long time, the orbit will have changed and there is no reason for which, on the new orbit, $< \dot{\Omega} >= 0$ should hold again. Therefore, the fact that a 3:2 resonance appears as an equilibrium in a model with fixed orbit means that, in a model where the orbit parameters are left free to vary, the 3:2 resonance will be stable for a long time, but not for an *infinite* time.

Of course, a very interesting point is the validation of the Darwin-Kaula theory. In particular, it would be very relevant to be able to give an estimate of the time scale of validity of the classical theories of bodily tides. In our language, the approximation by Darwin consists in the assumption that the degrees of freedom corresponding to the configuration of the deformable body adapt themselves with a delay given by the phase lags. It is very natural to think of this problem as an application of perturbation theory. We are currently working at the connection of our model with the Darwin-Kaula theories and what we are trying to prove is that our model reduces to the one by Darwin and Kaula at the second order in perturbation theory, the small parameter being the kinetic energy associated to the bodily deformations.

Chapter 2

Asymptotic behavior of a satellite

In this chapter we will refer to a dynamical system consisting of:

(i) a pointlike mass M (which we will sometimes call "planet"), whose space coordinates are fixed;

(ii) an elastic body with internal friction, of any shape, free to move in space (and to orbit the pointlike mass M); we will call this extended body "satellite".

In order to deal with the point (ii), we must set our study into the context of the theory of elasticity.

2.1 The setting

We will use the Lagrangian (or material) description of elasticity. In this approach one defines the so called material space Ω, which is essentially an abstract realization of the elastic body in some reference configuration. We denote by m the mass of the elastic body, i.e.

$$m = \int_{\Omega} \rho_0(\mathbf{x}) d^3\mathbf{x} \ , \tag{2.1.1}$$

where $\rho_0(\mathbf{x}_0)$ is the density of the elastic body at the material point $\mathbf{x}_0$, in the reference configuration.

The configuration of the body is a map $\zeta : \Omega \to \mathbb{R}^3$, which gives the position in space of the point $\mathbf{x} \in \Omega$.

Following the classical theory of elasticity, we define the displacement vector field

$$u(\mathbf{x}) := \zeta(\mathbf{x}) - \mathbf{x} \ , \tag{2.1.2}$$

which represents the displacement of each material point from the position it occupies in the reference configuration. Clearly, the displacement vector field cannot be identified with the deformation of the body, since, for instance, a rigid translation or rotation of the body produces a nonzero displacement. However, the displacement vector field contains all the information about the position of each material point and, therefore, it contains all the information about the deformation of the body.

In the linear theory of elasticity, the deformation of an elastic body is described through the strain tensor ε_{ij}, which arises in the following way. In the neighborhood of a material point $\mathbf{x}_0$, the displacement is given by

$$u(\mathbf{x}) = u(\mathbf{x}_0) + \nabla u(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) \tag{2.1.3}$$

in the linear approximation. In order to give a physically relevant description of the local deformation, it is useful to split the gradient of the displacement $\nabla u(\mathbf{x}_0)$ into its symmetric and skew-symmetric parts. Precisely, we set

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.1.4}$$

and

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) , \tag{2.1.5}$$

so that

$$\frac{\partial u_i}{\partial x_j} = \varepsilon_{ij} + \omega_{ij} , \tag{2.1.6}$$

where all the tensors and partial derivatives are understood to be evaluated at $\mathbf{x}_0$. In this decomposition, ω_{ij} is the skew-symmetric tensor which describes local rotation, while the symmetric part ε_{ij} is the strain tensor, which describes the local deformation.

Then one needs to describe the forces acting within the elastic body. These forces are of two types: internal tractions and body forces. The tool for describing internal tractions is given by the stress tensor σ_{ij}. Its physical meaning is that $\sigma_{ij}(\mathbf{x}_0)$ is the i-th component of the internal traction acting upon the plane passing through $\mathbf{x}_0$ with normal vector $\mathbf{e}_j$, i.e. the j-th vector of the canonical basis of $\mathbb{R}^3$.

Finally, we denote by $\mathbf{f}$ the vector field of body forces per unit mass acting upon the elastic body.

Imposing the conservation of linear momentum, one gets

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \rho f_i + \frac{\partial}{\partial x_j}\sigma_{ij} \ , \tag{2.1.7}$$

which is the general equation of motion in the Lagrangian description of continuum mechanics. Here, $\rho(\mathbf{x})$ is the actual density at the point $\mathbf{x}$ when the body is deformed. Because of the conservation of mass, ρ is a function of the configuration through

$$\rho(\mathbf{x}) = \frac{\rho_0(\mathbf{x})}{\det\frac{\partial\zeta}{\partial\mathbf{x}}(\mathbf{x})} \ . \tag{2.1.8}$$

These equations, of course, are largely underdetermined unless one specifies:

(i) the relation between the displacement vector u and the stress tensor σ;

(ii) the boundary conditions on the surface of the elastic body.

2.1.1 Constitutive relations

Extended bodies of different materials behave in a different way when a stress is applied. In particular, varying the material, the same load of applied stress can produce different deformations. Such mechanical properties are specified by the so-called constitutive relations, which give the connection between the stress tensor $\sigma(= \sigma(\mathbf{x}))$ and the strain tensor ε.

In the theory of linear elasticity, the stress tensor is assumed to be a linear function of the strain, so that, in the purely elastic case, we have

$$\sigma = \sigma_{el} = \mathbf{B}\varepsilon \ , \tag{2.1.9}$$

where $\mathbf{B}$ is a linear operator, which may depend on the material point $\mathbf{x}$ if the body is not homogeneous.

When internal friction is considered, one has to take into account viscous effects and the stress tensor is no more a function of the strain only, since it depends also on the time derivative of the strain $\dot{\varepsilon}$. As for the elastic stress, in the linear theories the viscous stress is given by

$$\sigma_{visc} = \mathbf{A}\dot{\varepsilon} \ , \tag{2.1.10}$$

where, again, $\mathbf{A}$ is a linear operator which may depend on the material point $\mathbf{x}$.

A simple possibility, when dealing with materials which exhibit both an elastic and a viscous behavior, is to assume that the total stress is simply the sum of the elastic and the viscous one, i.e.

$$\sigma = \sigma_{el} + \sigma_{visc} = \mathbf{B}\varepsilon + \mathbf{A}\dot{\varepsilon} \ . \tag{2.1.11}$$

Actually, in order to prove our result, we need not assume that (2.1.11) holds. Instead, we will make the following assumption.

Assumption 1. *At every material point* $\boldsymbol{x}$*, the stress tensor* σ *is a function of the strain tensor* ε *and of its time derivative* $\dot{\varepsilon}$*:*

$$\sigma = F_{\boldsymbol{x}}(\varepsilon, \dot{\varepsilon}) \ . \tag{2.1.12}$$

Moreover, for all fixed $\boldsymbol{x} \in \Omega$*, the function* $g_{\boldsymbol{x}}$ *defined by*

$$g_{\boldsymbol{x}}(\varepsilon) := F_{\boldsymbol{x}}(\varepsilon, 0) \tag{2.1.13}$$

is invertible.

Remark 2.1.1. *Assumption 1 is satisfied if one assumes the constitutive relation* (2.1.11)*, provided that the linear operator* $\boldsymbol{B}$ *is invertible.*

Remark 2.1.2. *Assumption 1 also means that we are neglecting the possible hereditary behavior of the extended body. In order to take into account such hereditary effects, one would have to add integral terms, which would make the stress tensor* σ *depend on the time history of the strain tensor* ε*.*

Substituting (2.1.12) into the equation of motion (2.1.7), and provided that a suitable expression of the body force $\mathbf{f}$ as a function of the body configuration is known, the equation of motion of the extended body becomes a partial differential equation. In the case under study, $\mathbf{f}$ is the force of gravitation, which is obviously a function of the body configuration, whose explicit expression will be given in (2.4.3).

2.1.2 Boundary conditions

In order to have a chance to obtain a well-posed problem, one has to add suitable boundary conditions to the equation (2.1.7).

For the study of our problem, the most natural thing is to impose the free-surface boundary condition. This means that, on the surface of the extended body, the component of the internal traction normal to the surface of the body vanishes.

Denoting by $\mathbf{n_x}$ the exterior normal to the surface $\zeta(\partial\Omega)$ at the point $\zeta(\mathbf{x}) \in \zeta(\partial\Omega)$, we therefore impose the following boundary condition.

For all $\boldsymbol{x} \in \partial\Omega$,

$$\sigma \cdot \mathbf{n_x} = 0 \ . \tag{2.1.14}$$

2.2 The Cauchy problem

In the previous section, we have introduced the equation of motion (2.1.7) of continuum mechanics, which, in particular, holds for the motion of an elastic body. The natural subsequent step would be that of searching for solutions to (2.1.7) (together with boundary and initial conditions) in a suitable function space. In order to discuss dynamics, one should in principle prove an existence and uniqueness theorem for the solutions to the Cauchy problem and, in order to use energy conservation (or dissipation) to prove dynamical properties, one should also prove that the dynamics is well-posed in the energy space, a fact which is in general unknown. To this end, we remark that, for a given state of the system, which is individuated in the phase space by the configuration of the body and by its velocity field, the energy is given by a suitable functional, which has the structure $E(u, \dot{u}) = E_{kin}(\dot{u}; u) + E_{pot}(u)$. The potential energy, in its turn, has the structure $E_{pot}(u) = E_g(u) + E_{sg}(u) + E_{el}(u)$, where E_g is the potential energy of the extended body in the gravitational field generated by M, E_g is the energy of self-gravitation and E_{el} is the elastic energy of deformation. In general, the elastic energy is a nonlinear function of the strain tensor $\varepsilon = \{\varepsilon_{ij}\}_{i,j=1}^3$:

$$E_{el}(u) = \int_\Omega f(\varepsilon) d^3\mathbf{x} \; . \tag{2.2.1}$$

In the linear theories of elasticity, this reduces to

$$E_{el}(u) = \int_\Omega \sum_{i,j,k,l=1}^{3} B_{ijkl}\varepsilon_{ij}\varepsilon_{kl} d^3\mathbf{x} \; , \tag{2.2.2}$$

where B_{ijkl} are the elements of the stiffness tensor.

In this thesis, we will not enter such a kind of mathematical problems, so we will simply assume these well-posedness properties.

Assumption 2. *There exists a function space X such that:*

(i) the Cauchy problem given by the equations

$$\begin{cases} \rho\frac{\partial^2 u_i}{\partial t^2} = \rho f_i + \frac{\partial}{\partial x_j}\sigma_{ij} & x \in \Omega \\ \sigma \cdot \boldsymbol{n_x} = 0 & x \in \partial\Omega \\ u(0) = u_0 \\ \dot{u}(0) = v_0 \end{cases} \tag{2.2.3}$$

admits a unique solution for all initial data $(u_0, v_0) \in X$*.*

(ii) The energy functional $E(u, v)$ *(as well as* $E_{kin}(\dot{u}; u)$*,* $E_g(u)$*,* $E_{sg}(u)$ *and* $E_{el}(u)$*) and its Lie derivative* $\frac{d}{dt}E(u, v)$ *along the flow of* (2.2.3) *are defined for all* $(u, v) \in X$ *and the inequality*

$$\frac{d}{dt}E(u, v) \leq 0 \tag{2.2.4}$$

is satisfied for all $(u, v) \in X$*.*

Moreover, in (2.2.4) *the equality holds if and only if the symmetric part of the gradient of* v *vanishes. This implies that* $\frac{d}{dt}E(u, \dot{u}) = 0$ *if and only if the time rate of change of the deformation vanishes, i.e. if and only if* $\dot{\varepsilon} = 0$*.*

2.3 LaSalle's invariance principle

In order to study the dynamics of the system, we will make use of the so-called LaSalle's invariance principle. LaSalle's principle is a refinement of the classical Lyapunov's theorem, which allows one to prove results of asymptotic stability in presence of a Lyapunov function E satisfying a nonstrict inequality of the type $\dot{E} \leq 0$.

In order to give a formulation of this principle, we now fix some notation and recall some basic facts and definitions.

Let

$$\dot{\mathbf{x}} = f(\mathbf{x}) \qquad \mathbf{x} \in X \tag{2.3.1}$$

be a system of differential equations. We denote by φ the flow of (2.3.1), i.e. $\varphi(t, \mathbf{x}_0)$ is the value, at the time t of the solution to (2.3.1) with initial datum $\mathbf{x}_0$. The flow is a priori well-defined only locally in time; however, there may be initial points for which the flow is well-defined for all times, or at least for all positive times. If an orbit is defined for all positive times, then one can investigate about the behavior of the orbit, as $t \to +\infty$, which involves the definition of ω-limit.

Definition 2.3.1. *Let γ be the orbit of* (2.3.1) *with initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$. A point $\boldsymbol{y}$ is said to be an ω-limit point of γ if there exists a sequence of times $t_n \to +\infty$ such that*

$$\lim_{n \to +\infty} \varphi(t_n, \boldsymbol{x}) = \boldsymbol{y} \ . \tag{2.3.2}$$

Definition 2.3.2. *The ω-limit set of an orbit γ is defined as the union of all the ω-limit points of γ.*

When trying to prove results of stability, one has to guarantee that the ω-limit of the considered orbits is non-empty. To this end, some compactness assumption is needed. Then, the classical version of LaSalle's principle may be enunciated as follows:

Theorem 2.3.3 (LaSalle's invariance principle)**.** *Let K be a compact subset of the phase space X. Suppose that E is a real-valued smooth function defined on K, whose Lie derivative satisfies $\dot{E}(\boldsymbol{x}) \leq 0$ for all $\boldsymbol{x} \in K$. Let M be the largest invariant set contained in $N := \left\{\boldsymbol{x} \in K | \dot{E}(\boldsymbol{x}) = 0\right\}$. Then the ω-limit of every orbit which remains within K for $t > 0$ is a non-empty subset of M, which implies that such an orbit is asymptotic to M.*

We remark that the fact that the ω-limit is an invariant set under the flow of the system of differential equations is a standard fact, since it is a simple consequence of the group property of the flow. LaSalle's principle guarantees that it must also be a set where the Lie derivative of the function E vanishes. If one thinks of E as the energy of a system with dissipation, the consequence of the invariance principle is that the dynamics will lead to an asymptotic situation where no dissipation is present.

Proof. (LaSalle's invariance principle) Let $\gamma := \{\varphi(t, \mathbf{x}_0) | t > 0\}$ be a forward orbit, contained in the compact set K. To begin with, we remark that the ω-limit of γ is non-empty. If t_n is a sequence of positive times diverging to $+\infty$, then, by the compactness of K, there exists a subsequence t_{n_k} such that $\mathbf{x}(t_{n_k})$ converges to some $\mathbf{x}_0 \in K$. Moreover, since compact sets are closed, the ω-limit of γ is a non-empty subset of K. Now, let $\mathbf{y}$ belong to the ω-limit of γ. Then, to prove that the ω-limit is an invariant set one must show that $\varphi(t, \mathbf{y})$ belongs to the ω-limit of γ, for all $t \in \mathbb{R}$. Now, since $\mathbf{y}$

belongs to the ω-limit of γ, there exists a sequence $t_n \to +\infty$ such that $\varphi(t_n, \mathbf{x}_0) \to \mathbf{y}$. But we have

$$\varphi(t, \mathbf{y}) = \varphi(t, \lim_{n\to+\infty} \varphi(t_n, \mathbf{x}_0)) = \lim_{n\to+\infty} \varphi(t + t_n, \mathbf{x}_0)\ .$$

Setting $s_n := t + t_n$ and observing that $s_n \to +\infty$, we have that $\varphi(t, \mathbf{y})$ belongs to the ω-limit of γ.

We still have to prove is that the ω-limit must be contained in N. Let $\mathbf{y}_0$ be a point of the ω-limit of γ. Then there exists a sequence $t_n \to +\infty$ such that $\varphi(t_n, \mathbf{x}_0) \to \mathbf{y}_0$. Now, let

$$c := E(\mathbf{y}_0) = \lim_{n\to+\infty} E[\varphi(t_n, \mathbf{x}_0)]\ .$$

Since $E[\varphi(t, \mathbf{x}_0)]$ is a time-nonincreasing function,

$$\lim_{n\to+\infty} E[\varphi(t_n, \mathbf{x}_0)] = c$$

implies

$$\lim_{t\to+\infty} E[\varphi(t, \mathbf{x}_0)] = c\ .$$

Therefore, for all $\mathbf{y}$ in the ω-limit of γ, $E(\mathbf{y}) = c$ holds. Hence, the ω-limit is an invariant set contained in a level set of the function E. Therefore, the Lie derivative of E must vanish at every point of the ω-limit, i.e. the ω-limit of γ is a subset of N. Since the ω-limit is an invariant set, it must be a subset of M.

Now, since $M \subset K$, this implies that the orbit is asymptotic to the set M. In fact, suppose by contradiction that there exist $\delta > 0$ and a sequence $t_n \to +\infty$ such that $\mathrm{dist}(\mathbf{x}(t_n), M) \geq \delta$. The sequence $\mathbf{x}(t_n)$ is contained in the compact set K, therefore the set Ω of accumulation points of $\mathbf{x}(t_n)$ is a nonempty subset of K. Since $\mathrm{dist}(\mathbf{x}(t_n), M) \geq \delta$, we have $\Omega \cap M = \emptyset$. But, reasoning exactly in the same way as above in the proof, one can show that $\Omega \subset M$, which leads to a contradiction.

This concludes the proof of the invariance principle.

$\square$

The classical version of LaSalle's principle can be slightly modified for our aims. We first give the following definition.

Definition 2.3.4. *Let S be an invariant subset of the phase space. We say that S is* ***stable*** *(in the future) if the following condition is satisfied.*
For all $\epsilon > 0$, there exists $\delta > 0$ such that, if $\mathrm{dist}(\boldsymbol{x}_0, S) < \delta$, *then* $\mathrm{dist}(\varphi(t, \boldsymbol{x}_0), S) < \varepsilon$ *for all (positive) times.*

If we replace the compactness assumption in the invariance principle by simply requiring that the orbit encounters the compact set K at some diverging sequence of positive times and we repeat the same proof as for the classical version, we get the following proposition.

Proposition 2.3.5. *Let K be a compact subset of the phase space X. Suppose that E is a real-valued smooth function defined on X, whose Lie derivative satisfies $\dot{E}(\boldsymbol{x}) \leq 0$ for all $\boldsymbol{x} \in X$. Let M be the largest invariant set contained in $N := \left\{\boldsymbol{x} \in X | \dot{E}(\boldsymbol{x}) = 0\right\}$. Let $\gamma := \{\varphi(t, \boldsymbol{x}_0) | t > 0\}$ be a forward orbit, such that $\varphi(t_n, \boldsymbol{x}_0)$ is contained in*

K for some sequence of positive times $t_n \to +\infty$. Then the ω-limit of γ is a non-empty subset of M.

The asymptotic stability, under this weaker compactness assumption, is recovered when the invariant set M is stable, in the sense of Definition 2.3.4.

Corollary 2.3.6. *Let K be a compact subset of the phase space X. Suppose that E is a real-valued smooth function defined on X, whose Lie derivative satisfies $\dot{E}(\boldsymbol{x}) \le 0$ for all $\boldsymbol{x} \in X$. Let M be the largest invariant set contained in $N := \left\{\boldsymbol{x} \in X | \dot{E}(\boldsymbol{x}) = 0\right\}$. Let $\gamma := \{\varphi(t, \boldsymbol{x}_0) | t > 0\}$ be a forward orbit, such that $\varphi(t_n, \boldsymbol{x}_0)$ is contained in K for some sequence of positive times $t_n \to +\infty$. If the set M is stable in the future, then γ is asymptotic to M in the future.*

Proof. Fix $\varepsilon > 0$. Then, by Definition 2.3.4, there exists $\delta = \delta(\varepsilon)$ such that, if $\mathrm{dist}(\mathbf{x}_0, M) < \delta$, then $\mathrm{dist}(\varphi(t, \mathbf{x}_0), M) < \varepsilon$ for all positive times. We apply Proposition 2.3.5 to the orbit γ and we have that there exist $\mathbf{y} \in M$ and a sequence $s_n \to +\infty$ such that $\varphi(s_n, \mathbf{x}_0) \to \mathbf{y}$, which implies $\mathrm{dist}(\varphi(s_n, \mathbf{x}_0), M) \to 0$. Therefore, there exists n_0 such that $\mathrm{dist}(\varphi(s_{n_0}, \mathbf{x}_0), M) < \delta(\varepsilon)$. This implies that $\mathrm{dist}(\varphi(t, \mathbf{x}_0), M) < \varepsilon$ for all $t > s_{n_0}$. □

2.4 Characterization of non-dissipating orbits

In this section, we come back to the study of the system constituted by a fixed pointlike mass M and an elastic body whose properties are described in the first section of this chapter.

We will consider solutions to (2.1.7), with boundary conditions and constitutive relations specified respectively by Assumption (2.1.14) and by Assumption (2.1.12).

In the equation (2.1.7), we have to specify the expression of the body force $\mathbf{f}$ as a function of the body configuration. In the case under study, the only external force acting on the deformable body is the gravitational force, which is the sum of the force exerted by the pointlike mass M and the force of self-gravitation, i.e. the force of gravity that on each portion of the deformable body is exerted by the rest of the body.

Denoting by G the gravitational constant, the potential generated by M at the material point $\mathbf{x}$ is given by

$$V_M(\mathbf{x}) := -\frac{GM}{|\zeta(\mathbf{x})|} \ , \tag{2.4.1}$$

while the potential of self gravitation is

$$V_{sg}(\mathbf{x}) := -\int_\Omega \frac{G\rho(\mathbf{y})}{|\zeta(\mathbf{y}) - \zeta(\mathbf{x})|} d^3\mathbf{y} \ , \tag{2.4.2}$$

where it is worth noting again that the density ρ is function of the body configuration through (2.1.8).

The body force $\mathbf{f}$ is the gradient of the gravitational potential and, therefore, is given by

$$\mathbf{f} = \mathbf{f}_M + \mathbf{f}_{sg} = \nabla(V_M + V_{sg}) \ . \tag{2.4.3}$$

Since the main result that we will obtain in the next section is a consequence of LaSalle's principle, it is crucial to give a characterization of those solutions to (2.1.7) such that there is no dissipation of energy, i.e. $\frac{d}{dt}E(u,\dot{u}) = 0$ along the solution.

What we prove in this section is that the non-dissipating condition can be fulfilled only if the pointlike mass M is immobile in the reference frame of the extended body.

The heuristic idea behind the result stated in this section is the following: if there is no dissipation, then it means that the deformation of the body does not change in time, i.e. the motion is rigid-like. Now, fix a reference frame co-moving with the extended body. Consider the forces acting on the body: in the body frame, the stress tensor is constant in time, and the self-gravitation force is also constant, since the motion is rigid-like. The fact that the motion is rigid-like suggests that also the gravitational force due to the body M should not vary in time (in the body frame), otherwise there would be some change in the deformation. Of course, this is not obvious and it is what one really has to check.

Theorem 2.4.1. *Let $u(t)$ be a solution to* (2.1.7)*, with constitutive relations as in Assumption 1 and body force given by* (2.4.3)*, such that*

$$\frac{d}{dt}E(u,\dot{u}) = 0 \ . \tag{2.4.4}$$

Then, in a reference frame co-moving with the extended body, the position of the pointlike mass M is constant in time.

Proof. First of all, we remark that, due to Assumption 2, item (ii), the relation (2.4.4) is equivalent to requiring that $\dot{\varepsilon} = 0$, i.e. at each material point the strain tensor is constant in time. Note that, due to Assumption 1, this implies that also the stress tensor σ is constant in time at each material point. Since the strain tensor is constant at each point, then the body moves in space like a rigid body. In order to clarify this point, we observe that, if $\dot{\varepsilon} = 0$, then the time derivative of the tensor of infinitesimal rotation $\dot{\omega}$ must be the same for all points of the deformable body. To prove this, fix a point $\mathbf{x}_0 \in \Omega$. Then we prove that

$$\dot{\omega}(\mathbf{x}) = \dot{\omega}(\mathbf{x}_0)$$

for all $\mathbf{x} \in \Omega$. In fact, let γ be a path connecting $\mathbf{x}_0$ to $\mathbf{x}$ in Ω. Then, the value of $\omega(\mathbf{x})$ can be reconstructed from $\omega(\mathbf{x}_0)$ through the integration of a suitable first order differential form on the path γ:

$$\begin{aligned} 2\omega_{ij}(\mathbf{x}) &= \frac{\partial u_i}{\partial x_j}(\mathbf{x}) - \frac{\partial u_j}{\partial x_i}(\mathbf{x}) = \\ &= 2\omega_{ij}(\mathbf{x}_0) + \int_\gamma \sum_{l=1}^{3} \frac{\partial^2 u_i}{\partial x_j \partial x_l} dx_l - \int_\gamma \sum_{l=1}^{3} \frac{\partial^2 u_j}{\partial x_i \partial x_l} dx_l \ . \end{aligned} \tag{2.4.5}$$

Through the definition of the strain tensor, we have

$$\frac{\partial u_i}{\partial x_l} = 2\varepsilon_{il} - \frac{\partial u_l}{\partial x_i} \ . \tag{2.4.6}$$

Differentiating with respect to x_j, we find:

$$\frac{\partial^2 u_i}{\partial x_j \partial x_l} = 2\frac{\partial \varepsilon_{il}}{\partial x_j} - \frac{\partial^2 u_l}{\partial x_j \partial x_i} \ . \tag{2.4.7}$$

Exploiting (2.4.7), we can rewrite (2.4.5) as

$$\begin{aligned} 2\omega_{ij}(\mathbf{x}) &= 2\omega_{ij}(\mathbf{x}_0) + \int_\gamma \sum_{l=1}^{3} \left(2\frac{\partial \varepsilon_{il}}{\partial x_j} - \frac{\partial^2 u_l}{\partial x_j \partial x_i}\right) dx_l + \\ &\quad - \int_\gamma \sum_{l=1}^{3} \left(2\frac{\partial \varepsilon_{jl}}{\partial x_i} - \frac{\partial^2 u_l}{\partial x_i \partial x_j}\right) dx_l = \\ &= 2\omega_{ij}(\mathbf{x}_0) + 2\int_\gamma \sum_{l=1}^{3} \left(\frac{\partial \varepsilon_{il}}{\partial x_j} - \frac{\partial \varepsilon_{jl}}{\partial x_i}\right) dx_l \ . \end{aligned} \tag{2.4.8}$$

Now, differentiating with respect to time and exploiting $\dot{\varepsilon} = 0$, we have

$$\dot{\omega}_{ij}(\mathbf{x}) = \dot{\omega}_{ij}(\mathbf{x}_0) \ . \tag{2.4.9}$$

The condition of rigid motion may be expressed by saying that for all times there exists a vector ν (function of time, but independent of the material point), which is the angular velocity of the body, such that, for any fixed $\mathbf{x}_0 \in \Omega$, the equation

$$\dot{\zeta}(\mathbf{x}) - \dot{\zeta}(\mathbf{x}_0) = \nu \wedge [\zeta(\mathbf{x}) - \zeta(\mathbf{x}_0)] \tag{2.4.10}$$

holds for all $\mathbf{x} \in \Omega$. Differentiating with respect to time, we obtain

$$\ddot{\zeta}(\mathbf{x}) - \ddot{\zeta}(\mathbf{x}_0) = \nu \wedge \left[\dot{\zeta}(\mathbf{x}) - \dot{\zeta}(\mathbf{x}_0)\right] + \dot{\nu} \wedge [\zeta(\mathbf{x}) - \zeta(\mathbf{x}_0)] \ . \tag{2.4.11}$$

Substituting (2.4.10) in (2.4.11), we get

$$\ddot{\zeta}(\mathbf{x})-\ddot{\zeta}(\mathbf{x}_0) = \nu\wedge\{\nu \wedge [\zeta(\mathbf{x}) - \zeta(\mathbf{x}_0)]\}+\dot{\nu}\wedge[\zeta(\mathbf{x}) - \zeta(\mathbf{x}_0)] \ . \quad (2.4.12)$$

Now, let us work in the physical space instead of the material one, denoting by ξ the Cartesian coordinates of the physical space. Denoting by a and a_0, respectively, the accelerations at ξ and ξ_0, the equation (2.4.12) above can be rewritten as

$$\ddot{\xi} - \ddot{\xi}_0 = \nu \wedge [\nu \wedge (\xi - \xi_0))] + \dot{\nu} \wedge (\xi - \xi_0)) \ . \quad (2.4.13)$$

This equation holds for all times. In particular, (2.4.13) implies that, for all times, the acceleration field is a linear function of the position in the physical space.

If we look back at the structure of equation (2.1.7), we notice that it is nothing else but the local form of Newton's law: the acceleration at each point (note that $a = \ddot{u}$) equals the total force (which is the sum of body forces and internal stresses) per unit mass acting on the same point. But this observation, together with (2.4.13), implies that for each fixed time, the total force per unit mass is a linear function of the position inside the body. In particular, this implies that the second (and higher order) differential with respect to the space variables of the total force per unit mass is identically zero.

In other words, for any fixed time, we have that

$$\frac{d^2}{d\xi^2}\left(f_i + \frac{1}{\rho}\frac{\partial}{\partial x_j}\sigma_{ij}\right) = 0 \ , \quad (2.4.14)$$

where the terms f_i and $\frac{\partial}{\partial x_j}\sigma_{ij}$ must be thought of as functions of the ξ variables. This is a geometric property of the field of forces acting upon the extended body, which is verified for all times and is independent of the reference frame. In fact a change of reference frame is not a generic change of coordinates: changing the reference frame corresponds to making a linear change of coordinates and the property of linearity of a vector field is conserved when applying a linear change of coordinates.

Now, since the motion of the body is rigid, we can fix a reference frame in which the extended body is immobile. We say that such a reference frame is co-moving with the body and we denote by $\chi = L(t)(\xi)$ the coordinates in the co-moving frame, $L(t)$ being the composition of a translation and a rotation. Now, recall that the body force is simply the sum of the external gravitational force and the force of self-gravitation. Then, (2.4.14) can be rewritten as

$$\frac{d^2}{d\chi^2}\left(f_{M\,i} + f_{sg\,i} + \frac{1}{\rho}\frac{\partial}{\partial x_j}\sigma_{ij}\right) = 0\ . \tag{2.4.15}$$

Notice that the condition of rigid motion implies that, in the co-moving frame, $f_{sg\,i}$, ρ and $\frac{\partial}{\partial x_j}\sigma_{ij}$ are constant in time. Therefore, we have also

$$\frac{d}{dt}\left[\frac{d^2}{d\chi^2}\left(f_{sg\,i} + \frac{1}{\rho}\frac{\partial}{\partial x_j}\sigma_{ij}\right)\right] = 0\ . \tag{2.4.16}$$

Then, differentiating (2.4.15) with respect to time and subtracting

(2.4.16), we have

$$\frac{d}{dt}\left(\frac{d^2}{d\chi^2}f_{M\,i}\right) = 0 \ . \tag{2.4.17}$$

What we still have to prove is that (2.4.17) implies that the point-like mass M must be immobile in the co-moving frame. To this end, we have to compute the second differential of the gravitational field generated by M. Since the gravitational field is (except for the sign) the first differential of the gravitational potential, what we actually have to do is to compute the third differential of the gravitational potential generated by M. In a system of spherical coordinates (r, θ, ϕ), centered at M, the potential has the expression

$$V_M(r, \theta, \phi) = -\frac{GM}{r} \ . \tag{2.4.18}$$

At first glance, one might think that calculating the third differential in spherical coordinates should require using the complicated expression of the third differential in spherical coordinates, but the spherical symmetry allows us to do a straightforward computation in Cartesian coordinates and deduce the expression in spherical coordinates. The expression of the gravitational potential in Cartesian coordinates x, y, z (centered at M) is

$$V_M(x, y, z) = -\frac{GM}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} \ . \tag{2.4.19}$$

Since the expression is symmetric in the three variables x, y, z, there are only three third-order partial derivatives that we have to compute, the other ones being obviously obtainable by symmetry. The computations yield

$$\frac{\partial^3 V_M}{\partial x^3}(x,y,z) = \frac{3GMx(2x^2-3y^2-3z^2)}{(x^2+y^2+z^2)^{\frac{7}{2}}} \tag{2.4.20}$$

$$\frac{\partial^3 V_M}{\partial x^2 \partial y}(x,y,z) = \frac{3GMy(4x^2-y^2-z^2)}{(x^2+y^2+z^2)^{\frac{7}{2}}} \tag{2.4.21}$$

$$\frac{\partial^3 V_M}{\partial x \partial y \partial z}(x,y,z) = \frac{15GMxyz}{(x^2+y^2+z^2)^{\frac{7}{2}}} \ . \tag{2.4.22}$$

The next step is simply to evaluate these partial derivatives in a point of the form $(x_0,0,0)$, so that x represents the radial direction and y,z represent any direction orthogonal to the radial one. We find

$$\frac{\partial^3 V_M}{\partial x^3}(x_0,0,0) = \frac{6GM}{{x_0}^4} \tag{2.4.23}$$

$$\frac{\partial^3 V_M}{\partial x \partial y^2}(x_0,0,0) = \frac{\partial^3 V_M}{\partial x \partial z^2}(x_0,0,0) = -\frac{3GM}{{x_0}^4} \ , \tag{2.4.24}$$

all the other partial derivatives being zero, when evaluated at $(x_0,0,0)$. By the spherical symmetry of the potential, we deduce the general expressions in spherical coordinates

$$\frac{\partial^3 V_M}{\partial r^3}(r,\theta,\phi) = \frac{6GM}{r^4} \tag{2.4.25}$$

$$\frac{\partial^3 V_M}{\partial r \partial \theta^2}(r,\theta,\phi) = \frac{\partial^3 V_M}{\partial r \partial \phi^2}(r,\theta,\phi) = -\frac{3GM}{r^4} \tag{2.4.26}$$

$$\frac{\partial^3 V_M}{\partial \theta^3} = \frac{\partial^3 V_M}{\partial \phi^3} = \frac{\partial^3 V_M}{\partial r^2 \partial \theta} = \frac{\partial^3 V_M}{\partial r^2 \partial \phi} = 0 \tag{2.4.27}$$

$$\frac{\partial^3 V_M}{\partial\phi\partial\theta^2} = \frac{\partial^3 V_M}{\partial\phi^2\partial\theta} = \frac{\partial^3 V_M}{\partial r\partial\theta\partial\phi} = 0 \ . \tag{2.4.28}$$

We have said that the third differential of the potential V_M must be constant in time at any point of the extended body, in the co-moving frame. Actually, it will be enough to impose the condition that along the motion the third differential of the potential is conserved at a fixed point in the extended body, which is the image through the configuration map $\zeta(t)$ of a fixed material point $\mathbf{x}_0$. Now, let us consider a reference frame with origin at the point $\zeta(\mathbf{x}_0)$ and axes oriented along the r, θ and ϕ directions. A priori such a reference frame is not necessarily co-moving with the extended body. The components of a vector X in this reference frame will be denoted by X_r, X_θ, X_ϕ.

The third-order differential of V_M is a trilinear form, which has an associated cubic form

$$\begin{aligned} C(X) \quad &:= \quad d^3V_M(\zeta(\mathbf{x}_0))(X,X,X) = \\ &= \quad \frac{3GM}{|\zeta(\mathbf{x}_0)|^4}\left[X_r\left(2{X_r}^2 - 3{X_\theta}^2 - 3{X_\phi}^2\right)\right] \end{aligned} \tag{2.4.29}$$

Now, the conservation of the third differential of V_M implies that, in the co-moving frame, also the cubic form C must be constant, i.e. given any vector $X_c(t)$ co-moving with the extended body,

$$\frac{d}{dt}C[X_c(t)] = 0 \tag{2.4.30}$$

must hold. This obviously implies that also the set of zeros of the cubic form must be conserved in the co-moving frame. Now, the set of zeros of C is

$$Z = \left\{ X \in \mathbb{R}^3 | X_r \left(2{X_r}^2 - 3{X_\theta}^2 - 3{X_\phi}^2 \right) = 0 \right\} \ ,$$

which is the union of a plane orthogonal to the radial direction and a circular cone, whose axis is oriented along the radial direction. This argument shows that the radial direction (i.e. the direction of the line joining the pointlike mass M with $\zeta(\mathbf{x}_0)$) must be fixed in the co-moving frame. In other words, we can choose a co-moving frame with origin at $\zeta(\mathbf{x}_0)$ in such a way that one of the axes is always oriented along the radial direction: we choose this axis to be oriented *from M to $\zeta(\mathbf{x}_0)$*. We denote the unit vector associated to this axis with $\mathbf{e}_r$, so that the unit vectors of the co-moving frame will be $(\mathbf{e}_r, \mathbf{e}_y, \mathbf{e}_z)$, $\mathbf{e}_y$ and $\mathbf{e}_z$ being chosen arbitrarily. In this way, the coordinates of M in the co-moving frame are $(-|\zeta(\mathbf{x}_0)|, 0, 0)$.

Now, observe that

$$C(\mathbf{e}_r) = \frac{6GM}{|\zeta(\mathbf{x}_0)|^4} \ . \tag{2.4.31}$$

This, together with (2.4.30), implies that $|\zeta(\mathbf{x}_0)|$ must be constant in time. Finally, this implies that the pointlike mass M is immobile in the co-moving frame, which ends the proof of the theorem. □

In accordance with LaSalle's principle, we define the non-dissipating manifold and the largest invariant set contained in it.

Definition 2.4.2. *The non-dissipating manifold is*

$$\mathcal{ND} := \left\{ (u, v) \in X | \frac{d}{dt} E(u, v) = 0 \right\} \ .$$

Definition 2.4.3. *We define $\mathcal{ND}_{inv}$ to be the largest invariant set contained in $\mathcal{ND}$.*

We remark that, if the pointlike mass M occupies a fixed position in the co-moving frame, then the extended body shows always the same face to M, which is exactly what happens when the satellite is in synchronous resonance.

Then, the consequence of Theorem 2.4.1 is that $\mathcal{ND}_{inv}$ is a union of orbits of constant radius such that the extended body moves rigidly along the orbit in synchronous resonance. In non-degenerate cases, $\mathcal{ND}_{inv}$ is expected to be made up of a single orbit. In the next chapter, we will provide an example of both non-degenerate and degenerate cases.

2.5 The "three outcomes" theorem

In this section we exploit LaSalle's principle in order to prove what we call the theorem of the "three outcomes". The meaning of the theorem is that, in a system made up of an elastic satellite with friction and a pointlike planet whose space coordinates are fixed, there are only three admissible outcomes for the final behavior of the satellite:

(i) the satellite is expelled to infinity;

(ii) the satellite falls on the planet;

(iii) the satellite is captured in synchronous resonance.

For the discussion of the present section, it is convenient to isolate the motion of the center of mass of the satellite from the rest of the information related to the body configuration. Therefore, we define the center of mass

$$\mathbf{X} = \frac{1}{m} \int_{\Omega} \zeta(\mathbf{x}) \rho(\mathbf{x}) d^3\mathbf{x} \tag{2.5.1}$$

and the "centered configuration"

$$v(\mathbf{x}) := \zeta(\mathbf{x}) - \mathbf{X} \ . \tag{2.5.2}$$

of course, knowing $\mathbf{X}$ and the centered configuration $v(\mathbf{x})$ is equivalent to knowing the whole configuration $\zeta(\mathbf{x})$. Therefore, the Cauchy problem (2.2.3) can be reformulated in terms of $\mathbf{X}$ and v. In this way we are decomposing the phase space as $\big(\mathbb{R}^3 \setminus \{0\}\big) \times \mathbb{R}^3 \times Y$, $\big(\mathbb{R}^3 \setminus \{0\}\big) \times \mathbb{R}^3$ being the space where position and velocity of $\mathbf{X}$ live and Y being defined as the phase space related to the centered configuration v.

In order to prove our result, we need some definitions and technical hypotheses. First of all, we need to deal with the case of the impact between the satellite and the planet M. We remark that the problem has a singularity, in the sense that the equations of motion are not defined if the position a point of the satellite coincides with the position of the planet M, i.e. if there exists a point $\mathbf{x}_0 \in \Omega$ such that $\zeta(\mathbf{x}_0) = 0$.

Definition 2.5.1. *A solution to the Cauchy problem* (2.2.3) *(with any initial datum) is said to be impacting the planet (in the future) M if for all $\varepsilon > 0$ there exists a time $t > 0$ such that* $\mathrm{dist}(\zeta(\Omega, t), 0) < \varepsilon$.

Then we make the following assumption about the existence time of solutions to (2.2.3).

Assumption 3. *The existence time of a solution to* (2.2.3) *is infinite (in the future) if and only if the solution is not impacting the planet.*

We observe that this assumption implies that, if an impact is there, it takes a finite amount of time for the impact to occur. Moreover, it rules out the possibility of the occurrence of any sort of blow up which might correspond to some fracture, disintegration, or, generally speaking, singularity formation in the configuration of the satellite.

Then the following proposition is immediate.

Proposition 2.5.2. *A solution is not impacting the planet if and only if there exists $\delta > 0$ such that* $\mathrm{dist}(\zeta(\Omega, t), 0) \geq \delta$ *for all times $t > 0$.*

The next assumption is a rather technical one: in fact, LaSalle's invariance principle applies to orbits contained in a compact set. At the same time, bounded subsets of infinite-dimensional spaces are not necessarily pre-compact. In our case, what we need to assume is a usual fact in viscous-type equations. A typical effect of viscosity is the damping of high-frequency modes, which leads to pre-compactness properties of orbits in the future. Such properties are usually deduced by performing estimates which make use of the explicit form of the equations of motion. In our case, since we are dealing with quite a general setting, and we have no explicit expressions for the constitutive equations, we assume the following property.

Assumption 4. *For any initial datum, the solution to* (2.2.3) *is such that the corresponding future orbit* $\{(v(t), \dot{v}(t))|t > 0\} \subset Y$ *of the centered configuration is pre-compact.*

Finally, we assume the following property about the energy of the satellite.

Assumption 5. *The functional*

$$F(u) := E_{sg}(u) + E_{el}(u)$$

is bounded below.

We remark that the above assumption is satisfied if, in particular, the satellite has an equilibrium configuration under the effect of elastic stresses and self-gravitation which globally minimizes the associated energy.

We are now ready to state the main theorem.

Theorem 2.5.3 (Theorem of the three outcomes)**.** *Let Assumptions 1, 2, 3, 4 and 5 be satisfied. Then, for a solution to* (2.2.3)*, one of the following three (future) scenarios must occur:*

(i) the trajectory of the center of mass $\boldsymbol{X}$ *is unbounded;*

(ii) the solution impacts the planet;

(iii) the solution is asymptotic to the non-dissipating invariant manifold $\mathcal{ND}_{inv}$.

Proof. The proof of the theorem is actually a simple application of LaSalle's invariance principle. We are going to prove that any future orbit which does not satisfy either (i) or (ii) must necessarily satisfy (iii).

Let therefore γ be the orbit corresponding to the solution to (2.2.3), for some given initial data. Assume that γ does not satisfy either (i) or (ii). Then, by Proposition 2.5.2, we can conclude that the future trajectory of $\mathbf{X}$ is bounded above and below, i.e. there exist $k, K > 0$ such that $k \leq |\mathbf{X}| \leq K$ for all times.

In order to conclude that the velocity $\dot{\mathbf{X}}$ is bounded, we first look at the form of the energy functional

$$E(u, \dot{u}) = E_{kin}(u, \dot{u}) + E_g(u) + E_{sg}(u) + E_{el}(u) \ .$$

Observe that the sum $E_{sg}(u) + E_{el}(u)$ is bounded below, by Assumption 5. Furthermore, $E_g(u)$ is bounded below because, by Proposition 2.5.2, there exists δ such that $\mathrm{dist}(\zeta(\Omega), 0) \geq \delta$ for all times. Then, by the non-increasing energy inequality (2.2.4), we can conclude that the kinetic energy $E_{kin}(u, \dot{u})$ is bounded above in the future. Then, by König's second theorem, the total kinetic energy must be greater or equal to the kinetic energy of the center of mass of the satellite. Therefore, the kinetic energy of the center of mass of the satellite must be bounded above in the future, which implies that $\dot{\mathbf{X}}$ is bounded above in the future.

Then, since the future orbit of $(\mathbf{X}, \dot{\mathbf{X}})$ is contained in a compact subset of $\left(\mathbb{R}^3 \setminus \{0\}\right) \times \mathbb{R}^3$ and the future orbit of the centered configuration $(v, \dot{v})$ is pre-compact, then the future orbit of $(u, \dot{u})$ is pre-compact in X. Therefore, we can apply LaSalle's invariance principle, which states that the solution is asymptotic to the non-dissipating invariant manifold $\mathcal{ND}_{inv}$, which concludes the proof of the theorem.

$\square$

2.5.1 Comments about the meaning of the theorem of the three outcomes

In order to understand the content of the theorem of the three outcomes, we should first remark what it does not say. First of all, the theorem in itself says nothing about the stability of the synchronous resonance. At the same time, if one is able to prove the orbital stability of the synchronous resonance (which, apart from the matter of giving a rigorous proof, is something which is strongly expected to be true), then the local asymptotic stability is a trivial consequence of the theorem.

Some comment is needed about the meaning of the three outcomes.

Let us first consider the outcome (i), i.e. the case of unbounded orbit. The fact that the trajectory of the center of mass of the satellite is unbounded does not necessarily mean that it tends to infinity. In fact, one cannot a priori rule out the possibility that, along with an unbounded trajectory of the center of mass, there exist $R > 0$ and $t_n \to +\infty$ such that $|\mathbf{X}(t_n)| < R$. Anyway, if one can prove the orbital stability of the synchronous resonance, then a simple application of 2.3.6 proves that an unbounded trajectory of the center of mass must actually tend to infinity. In fact, under the assumption of orbital stability, one proves that, if there were $R > 0$ and $t_n \to +\infty$ such that $|\mathbf{X}(t_n)| < R$, then the orbit should be asymptotic to the synchronous resonance, which cannot be the case if the orbit is unbounded.

The outcome (ii) has the clear meaning of a planet-satellite collision and does not need any further explanation.

The outcome (iii) is the one we informally refer to as asymptotically being trapped into the synchronous resonance, since the characterization explained in Section 2.4 shows that the only possible non-dissipating behavior of the satellite is that of synchronous resonance, in the sense that the satellite always shows the same face to the planet, revolving about it at a fixed distance. Actually, in the least degenerate case, where the non-dissipating invariant manifold $\mathcal{ND}_{inv}$ is made up of only one orbit, the outcome (iii) clearly implies asymptoticity to such an orbit; however, in presence of very degenerate non-dissipating invariant manifolds, the outcome (iii) could a priori not result in asymptoticity to a synchronous resonant orbit. In order to perform some more accurate analysis of the manifold $\mathcal{ND}_{inv}$, one should add some assumptions about the structure of the satellite. However, in the next chapter we analyze, in the planar approximation (which we will specify in the next chapter), the two relevant cases of a triaxial satellite and of a satellite with spherical symmetry. In the triaxial case, the non-dissipating invariant manifold turns out to be completely non-degenerate; in the spherically symmetric case, the non-dissipating invariant manifold turns out to be a slightly degenerate one, in a sense that we will discuss later. In such a slightly degenerate situation, the outcome (iii) can still be referred to as asymptotically approaching the synchronous resonance.

A remarkable fact is that, anyway, the discussion of the present chapter excludes the possibility that other periodic orbits exist, different from the synchronous resonant ones. This may seem quite surprising, since, for instance, some celestial bodies are known to be trapped in spin-orbit resonances different from the synchronous one (think, for example, of the 3:2 of Mercury, seen as a satellite of the Sun). At the same time, from our point of view, it is natural to think that any periodic orbit different from the synchronous resonant one would cause some periodic deformation of the satellite. Such a periodic deformation would, in turn, produce some periodic dissipation.

But one cannot continue dissipating periodically the same amount of energy for an infinite time, since this would lead the energy to $-\infty$, implying that a collision is going to occur. Our interpretation is that situations like the 3:2 resonance of Mercury are very likely to be metastable, and it would be very interesting to investigate further in this direction. For instance, one could try to give an estimate of the time of such a metastability: it would not be so surprising if such a time were of the same order of magnitude as the estimated lifespan of the Solar system.

Chapter 3

Orbital stability

In this chapter, as we did in the previous one, we will deal with a dynamical system made up of a a pointlike mass M (the "planet"), whose space coordinates are fixed at the origin of the Euclidean space $\mathbb{R}^3$, and an elastic body (the "satellite") with internal friction, subject to the force of gravity exerted by M.

Our aim here is to prove the orbital stability of the synchronous resonance. We will give the proof of the stability in two cases:

(i) triaxial satellite;

(ii) satellite with spherical symmetry.

The proof of the orbital stability will be done under some approximations. Namely, we will make the planar approximation (which we will explain later) and we will truncate the expansion of the gravitational potential at the quadrupole terms, neglecting higher order effects.

In both cases, we make the convenient assumption that the elasticity moduli of the satellite are very large.

Before making some kinematic considerations, let us say what we mean by “triaxial satellite”. To this end, we have to recall some basic facts about moments and axes of inertia.

We first need to recall what the matrix of inertia is. In principle, one may evaluate the matrix of inertia of an extended body with respect to any point in space; however, we will always refer to the matrix of inertia evaluated with respect to the center of mass.

Let B be an extended body which occupies a volume V in the three-dimensional Euclidean space. Fix a Cartesian frame of reference (x, y, z), with origin in the center of mass of B and let $\rho(x, y, z)$ be the density function of B. Then the matrix of inertia of B is a symmetric matrix

$$\begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{bmatrix}$$

where

$$I_{11} = \int\int\int_V \rho(x, y, z)(y^2 + z^2)dxdydz \tag{3.0.1}$$

$$I_{22} = \int\int\int_V \rho(x, y, z)(x^2 + z^2)dxdydz \tag{3.0.2}$$

$$I_{33} = \int\int\int_V \rho(x,y,z)(x^2+y^2)dxdydz \tag{3.0.3}$$

$$I_{12} = -\int\int\int_V xy\rho(x,y,z)dxdydz \tag{3.0.4}$$

$$I_{13} = -\int\int\int_V xz\rho(x,y,z)dxdydz \tag{3.0.5}$$

$$I_{23} = -\int\int\int_V yz\rho(x,y,z)dxdydz\ . \tag{3.0.6}$$

Since the matrix of inertia is a real symmetric matrix, it has real eigenvalues. The eigenvalues of the matrix of inertia are the *principal moments of inertia* of B. If the three eigenvalues are distinct, then the directions of the three associated eigenvectors are well defined and individuate the *principal axes of inertia* of B.

As we will show later, the principal moments and axes of inertia are very relevant objects in the study of our problem. On the one side, the kinematic study of extended bodies rotating in space involves the concept of moment of inertia. On the other side, which is more specific to our problem, if one expands in multipoles the gravitational interaction of an extended body with a pointlike mass and neglects terms beyond quadrupole, the only relevant parameters of the extended body (apart from its mass and the position of its center of mass) are found to be the principal moments of inertia and the directions of the principal axes of inertia.

A *rigid* body is called triaxial if its three moments of inertia are distinct. In our framework, we deal with a body (the satellite) which is not rigid. When we talk about triaxiality of the satellite, we mean that the satellite is triaxial when it reaches its configuration of equilibrium between the elastic and self-gravitating forces. In other words, the satellite is triaxial if the three eigenvalues of the matrix of inertia are distinct when the satellite is in such an equilibrium configuration.

In the triaxial case, the orbital stability of the synchronous resonance is a consequence of the orbital stability for a triaxial *rigid* body, the deformable case being a small perturbation of the rigid one.

In the spherically symmetric case, instead, there is no orbital stability of the synchronous resonance for a rigid body, and such a stability appears as a consequence of the tidal deformation of the satellite.

For this reason, the proof of orbital stability will be quite straightforward in the triaxial case, while it will require a careful and rather technical kinematic analysis in the spherically symmetric case.

3.1 General setting and global rotations

Let us now recall some notation we have already used in the previous chapter, only to fix ideas. Let Ω be the material space and let $\zeta : \Omega \to \mathbb{R}^3$ be the body. In this chapter *we assume that, in the reference configuration $\zeta_0(\boldsymbol{x}) = \boldsymbol{x}$, elastic forces and forces of self-gravitation are in equilibrium in the satellite.* As in the previous chapter, we denote by $\rho_0 : \Omega \to \mathbb{R}$ the density function in the reference configuration and by $\rho : \Omega \to \mathbb{R}$ the density function in a generic configuration ζ, which is related to ρ_0 by (2.1.8). Without loss of generality, we assume that the center of mass in the reference configuration is at the origin, i.e.

$$\int_\Omega \mathbf{x}\rho_0(\mathbf{x})d^3\mathbf{x} = 0 \ . \tag{3.1.1}$$

The map ζ describes both the deformation of the body and its position and rotation in space, so it is natural to try to decompose ζ into a rigid translation, a rigid rotation and an internal deformation. As in the previous chapter, we define the center of mass of the body by

$$\mathbf{X} = \frac{1}{m}\int_\Omega \zeta(\mathbf{x})\rho(\mathbf{x})d^3\mathbf{x} \ , \tag{3.1.2}$$

and decompose the configuration vector field ζ as

$$\zeta(\mathbf{x}) = \mathbf{X} + v(\mathbf{x}) \ , \tag{3.1.3}$$

where v is such that

$$\int_\Omega v(\mathbf{x})\rho(\mathbf{x})d^3\mathbf{x} = 0 \ . \tag{3.1.4}$$

Denote by $\mathcal{C}$ the space of the v's such that (3.1.4) holds.

Remark 3.1.1. *The space $\mathcal{C}$ is an infinite dimensional function space, so in order to discuss dynamics one should introduce a suitable norm in it, prove an existence and uniqueness theorem for the solutions to the Cauchy problem and, in order to use energy conservation* *(or dissipation) to prove dynamical properties, one should also prove that dynamics is well posed in the energy space, which is in general unknown [26].*

Here, we do not want to enter such a kind of mathematical problems, so we will simply assume existence and uniqueness for the solutions to the Cauchy problem and well-posedness of the dynamics in the energy space.

Now the body configuration is uniquely parameterized through the position $\mathbf{X}$ of the center of mass and the function $v \in \mathcal{C}$. The reference configuration corresponds to $\mathbf{X}_0 = 0, v_0(\mathbf{x}) = \mathbf{x}$.

Then we would like to factor out rotations in a way similar to translations, however this requires a careful discussion. The point is that it is clear what it means to rotate a body, but it is not clear how to say that a deformation does not rotate the body: as we will see, this is not a well defined concept.

To understand this point, we recall the standard analysis of the local deformation in linear elasticity theory, as explained in the previous chapter. Fix a point $\mathbf{x}_0 \in \Omega$. Then, the displacement vector field is defined by $u(\mathbf{x}) := \zeta(\mathbf{x}) - \mathbf{x}$. The gradient $\nabla u(\mathbf{x}_0)$ is decomposed as the sum of its symmetric part ε and its skew-symmetric part ω. Then, *at the point* $\boldsymbol{x}_0$, the local deformation is described by the strain tensor ε, while the skew-symmetric part of the gradient ω describes the local rotation. Thus, at a local level, the separation between deformation and rotation is well-defined and completely standard.

On the other hand, when considering the global configuration, the situation is more complicated, because it is not trivial at all to answer the following question. *Let the configuration ζ be assigned. Is the corresponding displacement vector field u a "pure deformation", in the sense that it does not "globally rotate the body", or is it given by the composition of a rotation of the body with some "pure deformation"?*

The answer to this question is easy if one considers only affine deformations, i.e. if one allows only displacements whose gradient is spatially constant. In fact, in this case, the rotation is simply described by the skew-symmetric part of the gradient of the displacement (evaluated at any point, since it is constant).

Nevertheless, if one wants to deal with general displacement fields, no obvious answer can be given to the previous question. One could try to give some reasonable definitions of what a "pure deformation" is. For instance, we could fix $\mathbf{x}_0 \in \Omega$ and say that u is a "pure deformation" if the displacement, *locally at* $\boldsymbol{x}_0$, does not contain any rotation, i.e. $\nabla u(\mathbf{x}_0)$ is symmetric. Anyway, this seems quite arbitrary; moreover, different choices of $\mathbf{x}_0$ would result in different definitions of "pure deformation".

In order to make it independent of the particular choice of a point, one could try to give the definition of "pure deformation" through an integral condition. This is done, for instance, in [31, 35], where such a definition (in the case of an incompressible homogeneous body) is given by imposing that the integral of the curl vanishes, i.e.

$$\int_{\Omega} \mathrm{curl}(u(\mathbf{x}))d^3\mathbf{x} = 0 \ . \tag{3.1.5}$$

But the choice of this definition is again arbitrary.

The real point is that there is no natural way of defining what it means that a displacement does not globally rotate the body. However, it is even trivial to explain what it means to rotate a body or to say that a configuration is obtained by rotating another configuration. Mathematically, this corresponds to the fact that there exists a group action $\mathcal{A}_1$ of the rotation group $SO(3)$ on the configuration space $\mathcal{C}$, defined by

$$\begin{aligned} \mathcal{A}_1 : \quad SO(3) \times \mathcal{C} &\to \mathcal{C} \\ (\Gamma, v) &\mapsto \Gamma v \ . \end{aligned} \tag{3.1.6}$$

This group action allows one to introduce a structure of principal fibre bundle in $\mathcal{C}$, the base manifold being the quotient $\mathfrak{M} := \mathcal{C}/SO(3)$. The elements of such a quotient manifold are what one could call "pure deformations".

The fact that the quotient of a manifold under a group action is still a manifold is not always true, so we have to recall some basic facts about group actions in order to justify our assertion.

Definition 3.1.2. *Let G be a group acting on a set X. The action is said to be **free** if the following condition is verified: if there exist $x \in X$ and $g \in G$ such that $gx = x$, then g is the identity.*

Definition 3.1.3. *The action of a topological group G on a topological space X is said to be* ***proper*** *if the mapping*

$$G \times X \to X \times X \ (g, x) \mapsto (gx, x)$$

is proper, i.e., inverses of compact sets are compact.

Theorem 3.1.4. *Let G be a topological group acting on a topological space M. If G is compact, then the action is proper.*

Theorem 3.1.5. *If a Lie group G acts freely and properly on a smooth manifold, then the quotient M/G is a smooth manifold.*

In our case, the fact that $SO(3)$ is a compact group and that the action $\mathcal{A}_1$ is obviously free implies that $\mathfrak{M}$ is a manifold.

As usual in this geometric context, it is useful to introduce coordinates in which a point of $\mathcal{C}$ is represented by an element of $SO(3)$ and an element of the base manifold. However a concrete representation of the elements of the quotient manifold can be obtained only locally, by introducing a local section of the bundle, namely by choosing a submanifold $\mathfrak{S}$ of $\mathcal{C}$, transversal to the group orbit. Now it is clear that there are infinitely many possible choices of the section of the bundle, which correspond to infinitely many admissible definitions of "pure deformations". Nonetheless, the physics is independent of the choice of the section $\mathfrak{S}$: a change in the choice of the section simply results in a change of coordinates. The fact that the section is only local is not a problem, as long as only small deformations are allowed.

Now, let $\mathcal{A}_1(v_0)$ be the orbit of the reference configuration under the action of the group $SO(3)$. Once the section has been chosen, it naturally induces a smooth one-to-one correspondence between a neighborhood of $\mathcal{A}_1(v_0)$ (obtained as the union of the orbits of the points of $\mathfrak{S}$ under the group action $\mathcal{A}_1$) and $SO(3) \times \mathfrak{S}$. Such a correspondence allows one to parameterize the space $\mathcal{C}$ of configurations through an element of the group $SO(3)$ and a point of the section $\mathfrak{S}$. In the context of classical mechanics, this is useful, since, due to the isotropy of space, the Lagrangian of the body is going to be independent from the element of $SO(3)$ and will depend only on the variable belonging to $\mathfrak{S}$.

We remark that the machinery we have just introduced describes a general fact, which is independent of all the assumptions we will make later in the paper. We can summarize the result of our discussion in the following theorem:

Theorem 3.1.6. *There exist a neighborhood $\mathcal{U} \subset \mathcal{C}$ of $\mathcal{A}_1(v_0)$ and a one-to-one smooth function $f : SO(3) \times \mathfrak{S} \to \mathcal{U}$ with the property that $f(\Gamma, w) = \mathcal{A}_1(\Gamma)w$. Therefore, Γ and w can be used as coordinates on the configuration space $\mathcal{C}$.*

3.2 The matrix of inertia as a function of the configuration

Since the satellite is deformable, in our setting the matrix of inertia of the satellite is not a fixed object, but it is a function of the body configuration. In a compact notation, the elements $\{I_{ij}\}_{i,j=1}^{3}$ of the inertia matrix I are given by the following formula:

$$I_{ij} = I_{ij}(v) := \mathbf{e}_i \cdot \int_\Omega v(\mathbf{x}) \wedge (\mathbf{e}_j \wedge v(\mathbf{x}))\rho(\mathbf{x})d^3\mathbf{x}\ . \tag{3.2.1}$$

In the rest of the chapter we will always denote with the double subscript the elements I_{ij} of the matrix of inertia, while the principal moments of inertia (i.e., the eigenvalues of the matrix I) will be denoted by I_1, I_2, I_3. Again, we remark that the principal moments of inertia are a function of the body configuration. The principal axes of inertia are also functions of the body configuration and we will denote them by $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$.

3.3 Planar restriction

We are going to study the dynamics of the satellite in the special case when the motion of the center of mass of the satellite is planar and the spin axis of the body is orthogonal to the plane of the orbit and coincides with one of the principal axes of inertia of the body. For this reason, we will restrict to "planar motion of the center of mass", "planar deformations" and "planar rotations". Precisely, we make the following assumptions. (Here and below we denote by $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ the vectors of the canonical base of $\mathbb{R}^3$.)

Assumption 6 (Planar motion of the center of mass). *We assume that $\boldsymbol{X}$ lies in the plane generated by $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$.*

Assumption 7 (Planar deformations). *The configuration is such that $\boldsymbol{e}_3$ is an eigenvector of I. We label the principal axes of inertia so that $\boldsymbol{u}_3 = \boldsymbol{e}_3$.*

In other words, we are assuming the matrix of inertia of the satellite to be of the form

$$I(v) = \begin{bmatrix} I_{11}(v) & I_{12}(v) & 0 \\ I_{12}(v) & I_{22}(v) & 0 \\ 0 & 0 & I_{33}(v) \end{bmatrix}, \tag{3.3.1}$$

so that $I_3(v) \equiv I_{33}(v)$.

Assumption 8 (Planar rotations). Γ *is a rotation about the $\boldsymbol{e}_3$-axis, i.e. it has the form*

$$\Gamma = \Gamma(\alpha) := \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.3.2}$$

3.4 The gravitational interaction

In this section we prove that, in the quadrupole approximation, the gravitational potential of the satellite, in the Newtonian field generated by M, depends only on:

(i) the mass of the satellite;

(ii) the position of the center of mass of the satellite;

(iii) the principal moments of inertia of the satellite;

(iv) the orientation of the principal axes of inertia of the satellite.

To start with, we fix some notation: (R, ψ) are the polar coordinates of the center of mass of the satellite in the plane of the orbit. We denote with γ the angle between the principal axis $\mathbf{u}_1$ and the line joining the planet to the center of mass of the satellite. Such a line is usually referred to as the *line of centers*. Then, we have the following result.

Proposition 3.4.1. *In the quadrupole approximation the gravitational potential energy of the body in the field generated by the mass M is given by*

$$V_g(\boldsymbol{X}, I_1, I_2, I_3, \gamma) := -\frac{GMm}{R} + \frac{GM}{R^3}\left[-I_1 + 2I_2 - I_3 + 3(I_1 - I_2)\cos^2\gamma\right] . \tag{3.4.1}$$

Proof. Let (x_1, x_2, x_3) be the Cartesian coordinates referred to the system with origin $\mathbf{X}$ and axes $\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3$. Then we introduce the spherical coordinates (r, ϑ, ϕ) of the generic point $\mathbf{P}$ in the satellite, defined by:

$$\begin{aligned} x_1 &= r\cos\vartheta\cos\phi && (3.4.2)\\ x_2 &= r\sin\vartheta\cos\phi && (3.4.3)\\ x_3 &= r\sin\phi\ . && (3.4.4) \end{aligned}$$

In this frame, the products of inertia I_{ij} $(i \neq j)$ vanish, i.e.

$$\begin{aligned} \int_{\zeta(\Omega)} \rho(\xi) r^2 \cos^2\phi \cos\vartheta \sin\vartheta d^3\xi &= 0 && (3.4.5)\\ \int_{\zeta(\Omega)} \rho(\xi) r^2 \cos\phi \sin\phi \cos\vartheta d^3\xi &= 0 && (3.4.6)\\ \int_{\zeta(\Omega)} \rho(\xi) r^2 \cos\phi \sin\phi \sin\vartheta d^3\xi &= 0\ , && (3.4.7) \end{aligned}$$

and the principal moments of inertia are given by

$$\begin{aligned} I_1 &= K_2 + K_3 && (3.4.8)\\ I_2 &= K_1 + K_3 && (3.4.9)\\ I_3 &= K_1 + K_2\ , && (3.4.10) \end{aligned}$$

where

$$\begin{aligned} K_1 &= \frac{1}{2}\int_{\zeta(\Omega)} \rho(\xi) r^2 \cos^2\phi \cos^2\vartheta d^3\xi && (3.4.11)\\ K_2 &= \frac{1}{2}\int_{\zeta(\Omega)} \rho(\xi) r^2 \cos^2\phi \sin^2\vartheta d^3\xi && (3.4.12)\\ K_3 &= \frac{1}{2}\int_{\zeta(\Omega)} \rho(\xi) r^2 \sin^2\phi d^3\xi\ . && (3.4.13) \end{aligned}$$

The gravitational potential energy V_g is:

$$V_g = -\int_{\zeta(\Omega)} \frac{GM\rho(\xi)}{|\xi|} d^3\xi = -\int_{\zeta(\Omega)} \frac{GM\rho(\xi)}{\sqrt{R^2 + r^2 - 2Rr\cos\eta}} d^3\xi \ , \tag{3.4.14}$$

where η is the angle between the line of centers and **XP**. Notice that the relation

$$\cos\eta = \cos\phi \cos(\vartheta + \gamma) \tag{3.4.15}$$

holds. Let us recall now how the multipole expansion arises. We have

$$\frac{1}{|\xi|} = \frac{1}{\sqrt{R^2 + r^2 - 2Rr\cos\eta}} = \frac{1}{R} \frac{1}{\sqrt{1 + \left(\frac{r}{R}\right)^2 - 2\left(\frac{r}{R}\right)\cos\eta}} \ . \tag{3.4.16}$$

In terms of the Legendre polynomials $P_n(z)$, one has

$$\frac{1}{\sqrt{1 + x^2 - 2xz}} = \sum_{n\geq 0} x^n P_n(z) \ . \tag{3.4.17}$$

In particular, we recall that

$$P_0(z) = 1$$

$$P_1(z) = z$$

$$P_2(z) = \frac{3z^2 - 1}{2} \ .$$

Taking the quadrupole approximation means to cut the sum at $n = 2$. We get

$$\frac{1}{|\xi|} = \frac{1}{R} \sum_{n\geq 0} \left(\frac{r}{R}\right)^n P_n(\cos\eta) \simeq \frac{1}{R} \left[1 + \frac{r}{R}\cos\eta + \left(\frac{r}{R}\right)^2 \frac{3\cos^2\eta - 1}{2}\right] , \tag{3.4.18}$$

so the potential energy becomes

$$V_g = -\int_{\zeta(\Omega)} \frac{GM\rho(\xi)}{R}\left[1 + \frac{r}{R}\cos\eta + \left(\frac{r}{R}\right)^2 \frac{3\cos^2\eta - 1}{2}\right] d^3\xi \ . \tag{3.4.19}$$

Here, the first term equals $-\frac{GMm}{R}$; the second one vanishes because $\mathbf{X}$ is the center of mass of the satellite; the third term, namely

$$V_t := -\int_{\zeta(\Omega)} \frac{GM\rho(\xi)(3\cos^2\eta - 1)r^2}{2R^3} d^3\xi \ ,$$

gives what we call the "tidal" potential energy. A brief manipulation shows that

$$\begin{aligned} V_t &= -\frac{GM}{R^3}\int_{\zeta(\Omega)} \frac{\rho(\xi)(3\cos^2\eta - 1)r^2}{2} d^3\xi = \\ &= -\frac{GM}{2R^3}\int_{\zeta(\Omega)} \rho(\xi)r^2[3\cos^2\phi\cos^2(\vartheta + \gamma) - 1]d^3\xi = \\ &= -\frac{3GM}{R^3}(K_1\cos^2\gamma + K_2\sin^2\gamma) + \frac{GM}{2R^3}\int_{\zeta(\Omega)} \rho(\xi)r^2 d^3\xi = \\ &= -\frac{3GM}{R^3}(K_1\cos^2\gamma + K_2\sin^2\gamma) + \\ &\quad + \frac{GM}{2R^3}\int_{\zeta(\Omega)} \rho(\xi)r^2[\sin^2\phi + \cos^2\phi(\sin^2\vartheta + \cos^2\vartheta)]d^3\xi = \\ &= -\frac{3GM}{R^3}(K_1\cos^2\gamma + K_2\sin^2\gamma) + \frac{GM}{R^3}(K_1 + K_2 + K_3) = \\ &= \frac{GM}{R^3}[-I_1 + 2I_2 - I_3 + 3(I_1 - I_2)\cos^2\gamma] \ . \end{aligned} \tag{3.4.20}$$

□

The proposition we have proved in this section is the reason why it is convenient to use the principal moments of inertia of the satellite as Lagrangian coordinates in the study of our dynamical system.

3.5 Triaxial satellite

As we said above, triaxiality means that the three principal moments of inertia of the satellite in its reference configuration v_0 are all distinct. Without loss of generality, we may assume that $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are the principal axes of inertia of the satellite in the reference configuration.

Therefore, in the present section, we will assume that the following assumption is satisfied.

Assumption 9 (Triaxiality)**.** *The reference configuration v_0 is such that $I_{12}(v_0) = 0$ and*

$$I_1(v_0) = I_{11}(v_0) < I_2(v_0) = I_{22}(v_0) < I_3(v_0) = I_{33}(v_0) \ . \tag{3.5.1}$$

Remark 3.5.1. *The assumption on $I_3(v_0)$ is actually useless, once we have done the planar restriction. However, this is the correct assumption that one should make in order to look for stability of the complete three-dimensional model, without the assumption of planarity. We leave the assumption in this form, in order to make physically relevant the situation that we are studying. Anyway, in the planar model, we could only assume $I_1(v_0) \neq I_2(v_0)$, the sign of the inequality between I_1 and I_2 being determined by a choice of notation.*

We are now ready to prove the orbital stability of the synchronous resonance for a triaxial satellite. The scheme of the proof is the following: we will first prove orbital stability for the conservative Lagrangian system that one obtains neglecting internal friction. Such a proof involves conservation of energy. Then we observe that, if one adds some extra dissipative force which makes energy a non-increasing function of time, then one still has orbital stability (in the future).

3.5.1 Lagrangian coordinates

By virtue of Assumption 8 and Theorem 3.1.6, we can use the following set of Lagrangian coordinates:

- the polar coordinates (R, ψ) of the center of mass of the satellite;
- the angle $\chi := \alpha - \psi$, describing the rigid rotation of the satellite, measured with respect to the line of centers;
- a set of Lagrangian coordinates for $w \in \mathfrak{S}$.

We start with the idea of using the principal moments of inertia and an angle describing the orientation of the principal axes of inertia as Lagrangian coordinates; however, there is the problem of singularity in the definition of the axes of inertia when two principal moments of inertia have the same value. In the triaxial case, since we need not exploit any further symmetry (as, instead, we will have to do in the spherically symmetric case), we can get rid of the problem of such a singularity by simply using the matrix elements $I_{ij}(w)$ as coordinates instead of the eigenvalues and eigenvectors of the matrix of inertia. In order to do this, we need the following assumption.

Assumption 10. *The functions*

$$I_{ij} : \mathfrak{S} \to \mathbb{R} \ ,$$

$$(i, j) = (1, 1), (2, 2), (3, 3), (1, 2)$$

are independent in a neighborhood of v_0*.*

Remark 3.5.2. *The previous assumption is satisfied, for instance, if for any* ij *there exists a deformation which modifies* I_{ij}*, leaving unaltered the other elements of the matrix* I*. The same assumption would not be satisfied if, for example, one added some additional constraint, like the incompressibility constraint. In that case, one would have to drop one degree of freedom.*

If the assumption of independence is satisfied, then one can complete the set of I_{ij}'s to a system of coordinates on the local section $\mathfrak{S}$, which is expressed by the following proposition.

Proposition 3.5.3. *There exist functions* $(z_1, z_2, \ldots)$*, with*

$$z_j : \mathfrak{S} \to \mathbb{R} \quad (j = 1, 2, \ldots)$$

such that $(I_{11}, I_{22}, I_{33}, I_{12}, z_1, z_2, \ldots)$ *is a set of smooth coordinates on* $\mathfrak{S}$*.*

Without loss of generality (just by applying a translation), we may also assume $z_j(v_0) = 0$.

Now that we have a set of Lagrangian coordinates, we seek for the expression of the Lagrangian function in these coordinates, which gives the equations of motion for the conservative system.

3.5.2 Potential energy

The potential energy is the sum of three terms: (1) the potential energy of the satellite in the gravitational field generated by M, (2) the elastic potential energy, (3) the energy of self-gravitation of the satellite.

Gravitational potential energy

Since we are using as coordinates the I_{ij}'s instead of the I_j's and the angle γ, we have to adjust, in terms of the new coordinates, the expression of the gravitational potential that was calculated in Proposition 3.4.1.

We obtain the following proposition.

Proposition 3.5.4. *In the quadrupole approximation the gravitational potential energy of the body in the field generated by the mass M is given by*

$$V_g(R, \chi, I_{11}, I_{22}, I_{33}, I_{12}) = -\frac{GMm}{R} + \tag{3.5.2}$$
$$+\frac{GM}{R^3}\left[-I_{11} + 2I_{22} - I_{33} + 3(I_{11} - I_{22})\cos^2\chi - 3I_{12}\sin(2\chi)\right] .$$

Proof. We follow the proof of Proposition 3.4.1. Everything works in the same way, except for the fact that we replace γ with χ and that, correspondingly, in the frame co-rotating with the satellite, the product of inertia I_{12} does not vanish. Then we have the thesis. $\square$

Elastic and self-gravitational energy

In our model, the elastic and self-gravitational forces play, in some sense, the same role, since the combined action of both forces tends to restore the reference configuration of the body and since they are both independent of the rotation angle χ. Therefore, we will not distinguish between the two corresponding potential energies. (With an abuse of language, we will refer to the sum of the elastic and self-gravitational potential energies simply as "elastic potential energy".) We will also assume that the satellite has very large moduli of elasticity, which will correspond to having a very small parameter ϵ. Using the notation $I = (I_{11}, I_{22}, I_{33}, I_{12})$, $z = (z_1, z_2, \ldots)$, we summarize these facts in the following assumption.

Assumption 11. *The elastic potential energy has the form*

$$V_e(I, z) = \frac{1}{\epsilon} V_0(I, z) \ , \tag{3.5.3}$$

where ϵ is a small parameter Q and V_0 has a nondegenerate minimum at $(I, z) = (I_0, 0)$, having set $I_0 := (I_{11}(v_0), I_{22}(v_0), I_{33}(v_0), I_{12}(v_0))$.

3.5.3 Kinetic energy

When evaluating the expression for the kinetic energy, it is convenient to express it in terms of the angle $\alpha = \chi + \psi$, which describes the actual speed of rotation of the satellite with respect to an inertial frame of reference. By König's second Theorem, the kinetic energy T can be written as the sum of two terms: the former is the kinetic energy of the center of mass

$$T_{cm} = \frac{m}{2}(\dot{R}^2 + R^2\dot{\psi}^2) \tag{3.5.4}$$

and the latter is the kinetic energy of the satellite with respect to its center of mass

$$T_r := T_r(\dot{\alpha}, \dot{I}, \dot{z}; I, z) \ . \tag{3.5.5}$$

Remark 3.5.5. *T_r is independent of α and due to the $\mathcal{A}_1$-invariance.*

Since the kinetic energy with respect to the center of mass is a quadratic form in the corresponding velocities, we will use the notation

$$T_r := \frac{1}{2}\sum_{i,k=1}^{+\infty} a_{ik}(I, z)\dot{q}_i\dot{q}_k \ , \tag{3.5.6}$$

where $q = (\alpha, I_{11}, I_{22}, I_{33}, I_{12}, z_1, z_2, \ldots)$. Observe that the coefficients $a_{ik}(I, z)$ are such that the quadratic form is positive definite on $\mathcal{C}$.

Lemma 3.5.6. *The coefficient $a_{11}(I, z)$ satisfies*

$$a_{11}(I, z) = I_{33} \ . \tag{3.5.7}$$

Proof. We have

$$v(\mathbf{x}) = \Gamma(\alpha) w(\mathbf{x}) \ . \tag{3.5.8}$$

Taking the derivative with respect to time, we get

$$\dot{v}(\mathbf{x}) = \frac{d\Gamma(\alpha)}{dt} w(\mathbf{x}) + \Gamma(\alpha)\dot{w}(\mathbf{x}) \ . \tag{3.5.9}$$

Therefore,

$$\begin{aligned} T_r &= \frac{1}{2}\int_\Omega [\dot{v}(\mathbf{x})]^2 \rho(\mathbf{x}) d^3\mathbf{x} = \frac{1}{2}\int_\Omega [\Gamma(-\alpha)\dot{v}(\mathbf{x})]^2 \rho(\mathbf{x}) d^3\mathbf{x} = \\ &= \frac{1}{2}\int_\Omega \left[\Gamma(-\alpha)\frac{d\Gamma(\alpha)}{dt} w(\mathbf{x}) + \dot{w}(\mathbf{x})\right]^2 \rho(\mathbf{x}) d^3\mathbf{x} = \\ &= \frac{1}{2}\int_\Omega [\omega \times w(\mathbf{x})]^2 \rho(\mathbf{x}) d^3\mathbf{x} + \qquad (3.5.10) \\ &\quad + \int_\Omega \langle \omega \times w(\mathbf{x}), \dot{w}(\mathbf{x})\rangle \rho(\mathbf{x}) d^3\mathbf{x} + \frac{1}{2}\int_\Omega [\dot{w}(\mathbf{x})]^2 \rho(\mathbf{x}) d^3\mathbf{x}, \end{aligned}$$

where ω is the angular velocity of the satellite, defined by

$$\omega \times (\cdot) = [\Gamma(-\alpha)][\frac{d}{dt}\Gamma(\alpha)](\cdot)$$

. Under our assumptions, $\omega = \dot{\alpha}\mathbf{u}_3$.

As the vector field $w(\mathbf{x})$ is independent of α, we observe that T_r is the sum of three integrals, the first of which gives the term in $\dot{\alpha}^2$, while the third one is a quadratic form in $(\dot{I}, \dot{z})$ and the second one gives mixed terms in $\dot{\alpha}$ and in the other velocities.

Therefore, one gets

$$a_{11}(I, z) = \int_\Omega \{\mathbf{u}_3 \times [\mathbf{x} + u(\mathbf{x})]\}^2 \rho(\mathbf{x}) d^3\mathbf{x} \ , \tag{3.5.11}$$

and it can easily be seen that this expression equals the moment of inertia related to the vertical axis, which concludes the proof of the lemma. □

3.5.4 The reduced Lagrangian

The Lagrangian of the system is given by

$$\mathcal{L} = T - V = T_{cm} + T_r - V_g - V_e \ , \tag{3.5.12}$$

where each of the terms is given by the expressions calculated in the previous sections. As a result, one gets

$$\begin{aligned} \mathcal{L} &= \frac{m}{2}\left(\dot{R}^2 + R^2\dot{\psi}^2\right) + T_r(\dot{\chi}+\dot{\psi}, \dot{I}, \dot{z}; I, z) + \frac{GMm}{R} + \\ &+ \frac{GM}{R^3}\left[I_{11} - 2I_{22} + I_{33} + 3(I_{22} - I_{11})\cos^2\chi + 3I_{12}\sin(2\chi)\right] + \\ &\quad -V_e(I, z; \epsilon) \ . \end{aligned} \tag{3.5.13}$$

Now, we observe that the Lagrangian does not depend on the cyclic coordinate ψ, so the total angular momentum

$$p := \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = mR^2\dot{\psi} + I_{33}(\dot{\chi} + \dot{\psi}) + 2\sum_{k=2}^{+\infty} a_{1k}(I, z)\dot{q}_k \ , \tag{3.5.14}$$

is a constant of motion. We can invert relation (3.5.14), to get the expression of $\dot{\psi}$ as a function of the other variables:

$$\dot{\psi} = \frac{p - I_{33}\dot{\chi} - 2\sum_{k=2}^{+\infty} a_{1k}(I, z)\dot{q}_k}{mR^2 + I_{33}} \ . \tag{3.5.15}$$

Then, we can drop one degree of freedom and study the reduced Lagrangian

$$\mathcal{L}^* = \mathcal{L} - \dot{\psi} \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \ , \tag{3.5.16}$$

where $\dot{\psi}$ must be thought of as a function of the other Lagrangian coordinates and velocities. After some calculations, we get

$$\mathcal{L}^* = T_2 + T_1 - \tilde{V} \ , \tag{3.5.17}$$

where

$$\begin{aligned}
T_2 &= \frac{m}{2}\dot{R}^2 + T_r(\dot{\chi}, \dot{I}, \dot{z}; I, z) - \frac{\left[I_{33}\dot{\chi} + 2\sum_{k=2}^{+\infty} a_{1k}(I,z)\dot{q}_k\right]^2}{2(mR^2 + I_{33})} \\
T_1 &= \frac{p\left[I_{33}\dot{\chi} + 2\sum_{k=2}^{+\infty} a_{1k}(I,z)\dot{q}_k\right]}{mR^2 + I_{33}} \\
\tilde{V} &= \frac{p^2}{2(mR^2 + I_{33})} - \frac{GMm}{R} + \\
&- \frac{GM}{R^3}\left[I_{11} - 2I_{22} + I_{33} + 3(I_{22} - I_{11})\cos^2\chi + 3I_{12}\sin(2\chi)\right] + \\
&+ V_e(J, z; \varepsilon) \ .
\end{aligned}$$

The conservative system given by the corresponding Euler-Lagrange equations has the conserved quantity

$$E := T_2 + \tilde{V} = \sum_{k=1}^{+\infty} \dot{y}_k \frac{\partial \mathcal{L}^*}{\partial \dot{y}_k} - \mathcal{L}^* \ , \tag{3.5.18}$$

where

$$y := (R, \chi, I, z)$$

and *the strict minima of $\tilde{V}$ are Lyapunov-stable equilibria of the reduced system.*

Let R_0 be a nondegenerate minimum of the function

$$V_{G0}(R) := -\frac{GMm}{R} + \frac{p^2}{2(mR^2 + I_{330})} - \frac{GM}{R^3}(I_{330} + I_{220} - 2I_{110}) \ , \tag{3.5.19}$$

where we have used the notation $I_{ij0} := I_{ij}(v_0)$.

Then we have the following

Lemma 3.5.7. *For any ϵ small enough, there exist $\bar{R}, \bar{\chi}, \bar{I}, \bar{z}$, s.t.*

(1) the point $\bar{y} := (\bar{R}, \bar{\chi}, \bar{I}, \bar{z})$ is a nondegenerate minimum of $\tilde{V}$.

(2) One has $(|\bar{R} - R_0|, \bar{\chi}, |\bar{I} - \bar{I}_0|, \bar{z}) = O(\epsilon)$.

(3) For all $\psi \in S^1$, in the configuration of the unreduced system corresponding to $(\bar{R}, \psi, \bar{\chi}, \bar{I}, \bar{z})$, the principal axis of inertia $\boldsymbol{u}_1$ is directed along the line of centers.

Remark 3.5.8. *A fixed point in the reduced system corresponds to a situation of synchronous resonance on a circular orbit in the unreduced system.*

Proof. We look for a minimum of $\tilde{V}$ in the domain $|I - I_0| \leq C\epsilon$ for some fixed C. Observe that, if ϵ is small enough, $|I - I_0| \leq C\epsilon$ implies both inequalities $I_1 < I_2 < I_3$ (we remind that $I_{12}(v_0) = 0$) and $I_{11} < I_{22} < I_{33}$, because of the triaxiality assumption. Moreover, if ϵ is small enough, also the inequality $|I_{12}| < I_{22} - I_{11}$ holds true.

First consider $\tilde{V}$ as a function of χ. Since $|I_{12}| < I_{22} - I_{11}$, $\tilde{V}$ has a nondegenerate minimum at

$$\chi = \chi(I) := \frac{1}{2} \arctan \frac{I_{12}}{I_{22} - I_{11}} \ . \tag{3.5.20}$$

Comparison with (3.4.1) shows that the angle $\chi(I)$ corresponds to $\gamma = 0$ in (3.4.1); this will imply the point (3) of the lemma. Consider now $\tilde{V}\big|_{\chi=\chi(I)}$; as a function of R it has a nondegenerate minimum at some point $R = R(I, z)$ fulfilling

$$|R(I, z) - R_0| \leq C\epsilon \ .$$

Consider now the restriction $\bar{V} = \bar{V}(I, z)$ of $\tilde{V}$ to the manifold $\chi = \chi(I)$, $R = R(I, z)$; since

$$\bar{V}(I, z) = \frac{1}{\epsilon} V_0(I, z) + O(1) \ , \tag{3.5.21}$$

such a function has a nondegenerate minimum close to zero.

Then the thesis follows. $\square$

Corollary 3.5.9. *The synchronous resonant circular orbit corresponding to $\bar{y}$ is orbitally stable for the Lagrangian system of equations*

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_k} = \frac{\partial \mathcal{L}}{\partial x_k} \ , \qquad (k = 1, 2, \ldots) \tag{3.5.22}$$

where $x = (R, \psi, \chi, I, z)$.

3.5.5 Dissipative dynamics

Now we modify the Euler-Lagrange equations of the previous section by introducing the effects of internal friction. To this end, we add terms $-f_k(\dot{x}, x)$ to the Euler-Lagrange equations (3.5.22), so that now we are going to study the system of equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_k} - \frac{\partial \mathcal{L}}{\partial x_k} = -f_k(\dot{x}, x) \ , \qquad (k = 1, 2, \ldots) \tag{3.5.23}$$

Moreover, since no dissipation acts on the orbital parameters, observe that the ψ-related term $f_2(\dot{x}, x)$ must be identically zero. Furthermore, since the dissipative forces are only function of the time evolution of the body configuration, all the f_k's must be independent of ψ and $\dot{\psi}$. As a consequence, one is again allowed to pass to the reduced system

$$\frac{d}{dt}\frac{\partial \mathcal{L}^*}{\partial \dot{y}_k} - \frac{\partial \mathcal{L}^*}{\partial y_k} = -\tilde{f}_k(\dot{y}, y) \ , \qquad (k = 1, 2, \ldots) \tag{3.5.24}$$

where the $\tilde{f}_k$ are obviously defined.

Now, proceeding as for the proof of energy conservation, we get the following lemma.

Lemma 3.5.10. *In the system of differential equations* (3.5.24)*, the Lie derivative of the energy is given by*

$$\frac{dE}{dt} = -\sum_{k=1}^{+\infty} \dot{y}_k \tilde{f}_k(\dot{y}, y) \ . \tag{3.5.25}$$

Proof. We have

$$\begin{aligned}
\frac{dE}{dt} &= \frac{d}{dt}\sum_{k=1}^{+\infty}\left(\frac{\partial\mathcal{L}^*}{\partial\dot{y}_k}\dot{y}_k - \mathcal{L}^*\right) = \\
&= \sum_{k=1}^{+\infty}\left[\left(\frac{d}{dt}\frac{\partial\mathcal{L}^*}{\partial\dot{y}_k} - \frac{\partial\mathcal{L}^*}{\partial y_k}\right)\dot{y}_k\right] = -\sum_{k=1}^{+\infty}\dot{y}_k\tilde{f}_k(\dot{y},y)\ .
\end{aligned} \tag{3.5.26}$$

□

Then, the property that the $\tilde{f}_k$-terms represent a dissipative force is summarized by the following

Assumption 12. *The functional form of the functions $\tilde{f}_k$ is such that*

$$\sum_{k=1}^{+\infty}\dot{y}_k\tilde{f}_k(\dot{y},y) \geq 0\ . \tag{3.5.27}$$

Then, a simple reasoning about the fact that energy is a non-increasing function of time, combined with Lemma 3.5.7, gives immediately the result of orbital stability of the synchronous resonance, which is expressed by the following theorem.

Theorem 3.5.11. *The point $(y = \bar{y}, \dot{y} = 0)$, with $\bar{y}$ as in Lemma 3.5.7, is a Lyapunov-stable (in the future) equilibrium for the reduced system of equations* (3.5.24)*.*

Corollary 3.5.12. *The synchronous resonant circular orbit corresponding to $\bar{y}$ is orbitally stable (in the future) for the complete system of equations* (3.5.23)*.*

3.6 Spherically symmetric satellite

In this section, we prove the orbital stability of the synchronous resonance for a satellite with spherical symmetry. When we say that the satellite is spherically symmetric, we mean the following things:

(i) the satellite, in the reference configuration $v_0(\mathbf{x}) = \mathbf{x}$, has a spherical shape, i.e. Ω is a three dimensional sphere centered at the origin;

(ii) the corresponding density function $\rho_0(\mathbf{x})$ is a purely radial function of $\mathbf{x}$;

(iii) the Lagrangian of the satellite (when ignoring the gravitational interaction with M) is invariant not only under the action $\mathcal{A}_1$ already introduced, but also under the action $\mathcal{A}_2$ defined in the following way:

$$\begin{aligned} \mathcal{A}_2 : \quad SO(3) \times \mathcal{C} &\to \mathcal{C} \\ (R, v) &\mapsto Rv \circ R^{-1} \ . \end{aligned} \tag{3.6.1}$$

The group action $\mathcal{A}_2$ has the following meaning. Imagine that the satellite is experiencing some deformation, which corresponds to a body configuration v. Then, applying $\mathcal{A}_2(R)$ to v corresponds to producing a configuration looks exactly like the previous one, except for the fact that the "direction" of the deformation inside the body has been rotated through the matrix R. We mean that if, for example, the initial configuration is an ellipsoid with some principal axes, then the second one is an ellipsoid with the same shape, but with axes which have been rotated inside the body. This is a true elastic deformation that involves dissipation.

Now we must face the following problems. With respect to the case of a triaxial satellite, now the situation is more degenerate, since, as we will prove later, the $\mathcal{A}_2$-invariance gives origin to a manifold of synchronous resonant orbits: heuristically, suppose you have one synchronous resonant orbit with a body configuration v; then, due to the $\mathcal{A}_2$-invariance, there exists a synchronous resonant orbit for every configuration of the type $\mathcal{A}_2(R)v$. At the same time, the presence of such an invariance gives a more symmetrical structure to the problem, which actually allows to cope with this slight degeneracy. However, when one tries to introduce coordinates adapted to this further symmetry, one has to cope with the fact that the group action $\mathcal{A}_2$ is not free, which makes non-trivial the fact of passing to the quotient. As a consequence, in order to pass to the quotient, we will have to exclude a singular set of configurations. Moreover it will turn out that, on the remaining "good" set, the adapted coordinates that we introduce form a 24-fold covering of the space of configurations. The most significant difficulties are precisely those related to the kinematic analysis which leads to the introduction of the Lagrangian coordinates.

3.6.1 Adapted coordinates

Since the action $\mathcal{A}_2$ is not free, we will consider also the action $\mathcal{A}_3$, combination of the actions $\mathcal{A}_1$ and $\mathcal{A}_2$, defined by

$$\mathcal{A}_3(R)v := \mathcal{A}_1(R)\mathcal{A}_2(R^{-1})v = v \circ R \tag{3.6.2}$$

and study the couple of actions $\mathcal{A}_1$ and $\mathcal{A}_3$. The advantage in introducing $\mathcal{A}_3$ is that such an action is free.

We introduce now an adapted set of coordinates in a neighborhood of the "identical" deformation

$$v_0(\mathbf{x}) = \mathbf{x} \tag{3.6.3}$$

(excluding however such a configuration). To this end we need to introduce a few objects:

(1) Define

$$\mathcal{C}_{\neq} = \{v \in \mathcal{C} | I_1 \neq I_2, I_1 \neq I_3, I_2 \neq I_3\} \tag{3.6.4}$$

and its complement

$$\mathcal{C}_{=} = \{v \in \mathcal{C} | I_i = I_j \text{ for some } i \neq j\} \ . \tag{3.6.5}$$

This is useful since the principal axes $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are uniquely determined in $\mathcal{C}_{\neq}$.

(2) Define

$$\mathcal{D} := \{v \in \mathcal{C} | I(v) \text{ is diagonal}\} \ . \tag{3.6.6}$$

We also define $\mathcal{D}_{\neq} := \mathcal{D} \cap \mathcal{C}_{\neq}$. Observe that $\mathcal{D}$ is a codimension 3 submanifold of $\mathcal{C}$, invariant under the action $\mathcal{A}_3$ (we will show in the proof of Lemma 3.6.5 that the action $\mathcal{A}_3$ leaves invariant the matrix of inertia) and observe that $v_0 \in \mathcal{D}$. Moreover, as we will prove in Lemma 3.6.5, on $\mathcal{D} \cap \mathcal{C}_{\neq}$ the action $\mathcal{A}_1$ is independent of the action $\mathcal{A}_3$, and is transversal to $\mathcal{D}$.

(3) Consider the group orbit $\mathcal{A}_3(SO(3))v_0 \subset \mathcal{D}$, and let $\mathcal{S} \subset \mathcal{D}$ be a codimension 3 (in $\mathcal{D}$) manifold transversal to such a group orbit. Actually we are interested in the restriction of such a section to a small neighborhood of v_0. We still denote by $\mathcal{S}$ such a *local* section. The existence of such an $\mathcal{S}$ is assured by the fact that the action $\mathcal{A}_3$ is free and therefore defines a foliation of $\mathcal{D}$.

(4) Finally define $\mathcal{F}$ to be the tube constituted by the orbits of $\mathcal{A}_1 \circ \mathcal{A}_2$ starting in $\mathcal{S} \cap \mathcal{C}_{\neq}$, namely

$$\mathcal{F} := \mathcal{A}_1(SO(3))\mathcal{A}_2(SO(3))(\mathcal{S} \cap \mathcal{C}_{\neq}) \ . \tag{3.6.7}$$

We remark that $\mathcal{F} = \mathcal{A}_1(SO(3))\mathcal{A}_3(SO(3))(\mathcal{S} \cap \mathcal{C}_{\neq})$.

Remark 3.6.1. *The eigenvalues I_j, as functions of the matrix elements $\{I_{ij}\}$ (and therefore of the configuration v), are smooth functions on $\mathcal{C}_{\neq}$; however, the first derivatives of the I_j's have a singu-*

larity at $\mathcal{C}_=$, therefore the I_j's can be used as Lagrangian coordinates only on $\mathcal{C}_{\neq}$.

Remark 3.6.2. *When restricting to the submanifold $\mathcal{D}$, the eigenvalues I_j coincide with the matrix elements on the main diagonal, and therefore they are obviously smooth functions of the configuration.*

Again, we make the following assumption.

Assumption 13. *We assume that the functions $I_j : \mathcal{D} \to \mathbb{R}$, $j = 1, 2, 3$ are independent in a neighborhood of v_0.*

In the rest of the section we will prove the following Theorem

Theorem 3.6.3. *There exist functions $(z_1, z_2, \ldots)$, with*

$$z_j : \mathcal{S} \to \mathbb{R} \quad (j = 1, 2, \ldots)$$

such that:

(i) $(I_1, I_2, I_3, z_1, z_2, \ldots)$ *is a set of smooth coordinates on $\mathcal{S}$.*

(ii) the map

$$\begin{array}{rcl} SO(3) \times SO(3) \times (\mathcal{S} \cap \mathcal{C}_{\neq}) & \to & \mathcal{F} \\ (\Gamma, R, I_1, I_2, I_3, z_1, z_2, \ldots) & \mapsto & \mathcal{A}_1(\Gamma)\mathcal{A}_2(R)w \end{array}$$

is a 24-fold covering of $\mathcal{F}$; here we denoted

$$w = (I_1, I_2, I_3, z_1, z_2, \ldots) \ .$$

Remark 3.6.4. *The number 24 arises as the order of the chiral octahedral group. More precisely it corresponds to the number of ways in which an oriented triple of orthonormal vectors can be rotated in such a way that the vectors lie on a triple of unoriented fixed orthogonal axes.*

Proof of Theorem 3.6.3

To begin with, we prove the existence of $(z_1, z_2, \dots)$ satisfying (i). Observe that $\mathcal{S}$ is a smooth submanifold of $\mathcal{C}$. Then, since $I_1(w)$, $I_2(w)$, $I_3(w)$ are independent functions, it is possible to complete the triple (I_1, I_2, I_3) to a local system of coordinates $(I_1, I_2, I_3, z_1, z_2, \dots)$ near v_0.

In the rest of the section, we will prove (ii). As a first step, we show that the two actions of $SO(3)$ on $\mathcal{C}_{\neq}$ are independent, which is implied by the following Lemma.

Lemma 3.6.5. *For any fixed $\hat{v} \in \mathcal{C}_{\neq}$, we consider two subspaces of $T_{\hat{v}}\mathcal{C}_{\neq}$, tangent to the group orbits $\mathcal{A}_1(SO(3))\hat{v}$ and $\mathcal{A}_3(SO(3))\hat{v}$, namely*

$$\mathcal{T}_1 := T_{\hat{v}}\mathcal{A}_1(SO(3))\hat{v} \qquad \mathcal{T}_3 := T_{\hat{v}}\mathcal{A}_3(SO(3))\hat{v} \ .$$

Then, $\mathcal{T}_1 \cap \mathcal{T}_3 = \{0\}$. Moreover, if $\hat{v} \in \mathcal{C}_{\neq} \cap \mathcal{D}$, then $\mathcal{T}_1$ is transversal to $\mathcal{D}$.

Proof. In order to prove the thesis, we start by showing that the action $\mathcal{A}_1$ rotates the matrix of inertia, while the action $\mathcal{A}_3$ leaves it invariant. We have

$$\begin{aligned} I_{ij}(\Gamma\hat{v}) &= \mathbf{e}_i \cdot \int_\Omega \Gamma\hat{v}(\mathbf{x}) \wedge (\mathbf{e}_j \wedge \Gamma\hat{v}(\mathbf{x}))\rho(\mathbf{x})d^3\mathbf{x} \\ &= \Gamma^{-1}\mathbf{e}_i \cdot \int_\Omega \hat{v}(\mathbf{x}) \wedge (\Gamma^{-1}\mathbf{e}_j \wedge \hat{v}(\mathbf{x}))\rho(\mathbf{x})d^3\mathbf{x} \ , \end{aligned} \tag{3.6.8}$$

which shows that the $I(\mathcal{A}_1(\Gamma)\hat{v})$ is the matrix of $I(\hat{v})$, just referred to a rotated basis. On the other hand, we have

$$I_{ij}(\hat{v} \circ R) = \mathbf{e}_i \cdot \int_\Omega \hat{v}(R\mathbf{x}) \wedge (\mathbf{e}_j \wedge \hat{v}(R\mathbf{x}))\rho(\mathbf{x})d^3\mathbf{x} \ . \tag{3.6.9}$$

Setting $\mathbf{y} = R\mathbf{x}$, we have

$$I_{ij}(\hat{v} \circ R) = \mathbf{e}_i \cdot \int_\Omega \hat{v}(\mathbf{y}) \wedge (\mathbf{e}_j \wedge \hat{v}(\mathbf{y}))\rho(\mathbf{y})d^3\mathbf{y} \ , \tag{3.6.10}$$

which means that the action of $\mathcal{A}_3$ leaves the matrix of inertia invariant. This implies

$$dI(\hat{v})v_3 = 0 \ \forall v_3 \in \mathcal{T}_3 \ , \tag{3.6.11}$$

while

$$dI(\hat{v})v_1 \neq 0 \ \forall v_1 \in \mathcal{T}_1 \setminus \{0\} \ , \tag{3.6.12}$$

from which the independence follows.

To get the transversality when $\hat{v} \in \mathcal{C}_{\neq} \cap \mathcal{D}$, we remark that $\mathcal{A}_1$ rotates the principal axes of inertia, then, since the three eigenvalues are distinct, it destroys the diagonal structure of I. □

Remark 3.6.6. *As an obvious corollary of Lemma 3.6.5, we also have that $\mathcal{A}_1$ and $\mathcal{A}_2$ are independent at any point $\hat{v} \in \mathcal{C}_{\neq}$, in the sense that $\mathcal{T}_1$ is transversal to the tangent space $\mathcal{T}_2 := T_{\hat{v}}\mathcal{A}_2(SO(3))\hat{v}$.*

Moreover, we observe that in $\mathcal{C}_{\neq}$ the three eigenvalues of the matrix of inertia are distinct, so the eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are well determined. Furthermore the dependence of the eigenvalues and eigenvectors on the configuration $v \in \mathcal{C}_{\neq}$ is smooth.

Now, we want to represent any configuration $v \in \mathcal{F}$ in the form

$$v = \mathcal{A}_1(\Gamma)\mathcal{A}_2(R)w \ , \quad w \in \mathcal{S} \cap \mathcal{C}_{\neq} \ . \tag{3.6.13}$$

Let us first represent any $v \in \mathcal{C}_{\neq}$ in the form

$$v = \mathcal{A}_1(\tilde{\Gamma})\tilde{w} \ , \quad \tilde{w} \in \mathcal{D}_{\neq} \ . \tag{3.6.14}$$

Proposition 3.6.7. *The map*

$$\begin{array}{rcl} \Pi : SO(3) \times \mathcal{D}_{\neq} & \to & \mathcal{C}_{\neq} \\ (\tilde{\Gamma}, \tilde{w}) & \mapsto & \mathcal{A}_1(\tilde{\Gamma})\tilde{w} \end{array}$$

is a 24-fold covering map.

The proof will make use of the following Theorem, which is an immediate corollary of [18], Proposition 1.40, p. 72.

Theorem 3.6.8. *If G is a finite group, acting freely on a Hausdorff space X, then the quotient map $X \to X/G$ is a covering map.*

Proof of Proposition 3.6.7. We observe that each $v \in \mathcal{C}_{\neq}$ has many distinct representations of the form (3.6.14): since the three principal moments of inertia are distinct from one another, the directions of the principal axes of inertia are well-determined, but the same is not true for what concerns their orientation; moreover, any of the principal axes may be labeled $\mathbf{u}_1$ as well as $\mathbf{u}_2$ or $\mathbf{u}_3$. In order to make this rigorous, consider the equation

$$\mathcal{A}_1(\tilde{\Gamma}_1)\tilde{w}_1 = \mathcal{A}_1(\tilde{\Gamma}_2)\tilde{w}_2 \ , \tag{3.6.15}$$

with $\tilde{w}_1, \tilde{w}_2 \in \mathcal{D}_{\neq}$. This implies

$$\tilde{w}_2 = \mathcal{A}_1(\tilde{\Gamma}_2^{-1}\tilde{\Gamma}_1)\tilde{w}_1 \ . \tag{3.6.16}$$

Therefore, since $\tilde{w}_1, \tilde{w}_2 \in \mathcal{D}_{\neq}$, the rotation $\tilde{\Gamma}_2^{-1}\tilde{\Gamma}_1$ must transform the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ to a set of unit vectors having the same directions. It is easy to see that the set of rotations satisfying this property is the subgroup of $SO(3)$ generated by the three rotations

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$R_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ .$$

Such a subgroup, which we will denote by O, is isomorphic to the group of the orientation preserving symmetries of the cube, which is a group of order 24, known as the *chiral octahedral group*. This argument shows that the possible representations of the form (3.6.14) are at most 24. On the other hand, for any $\tilde{w} \in \mathcal{D}_{\neq}$ and $\tilde{\Gamma} \in SO(3)$, we have that the expression

$$\mathcal{A}_1(\tilde{\Gamma}\Gamma_O)[\mathcal{A}_1(\Gamma_O^{-1})\tilde{w}] \tag{3.6.17}$$

yields 24 different representations of the same configuration, as Γ_O varies within the group O. Therefore, each configuration $v \in \mathcal{C}_{\neq}$ has exactly 24 distinct representations of the form (3.6.14) and a natural identification arises between $\mathcal{C}_{\neq}$ and $(SO(3)\times\mathcal{D}_{\neq}/O$, where the action of O on $SO(3) \times \mathcal{D}_{\neq}$ is defined by

$$[\Gamma_O, (\tilde{\Gamma}, \tilde{w})] \mapsto (\tilde{\Gamma}\Gamma_O, \mathcal{A}_1(\Gamma_O^{-1})\tilde{w}) \ . \tag{3.6.18}$$

Now, applying Theorem 3.6.8 with $X = SO(3) \times \mathcal{D}_{\neq}$ and $G = O$, we get the thesis. □

The above Proposition given as a global statement applies also to a small tube of orbits originating in $\mathcal{S}$. Precisely, define $\mathcal{T} := \mathcal{A}_3(SO(3))(\mathcal{S} \cap \mathcal{C}_{\neq})$: then we have

Corollary 3.6.9.

$$\begin{aligned} \Pi : SO(3) \times \mathcal{T} &\to \mathcal{F} \\ (\tilde{\Gamma}, \tilde{w}) &\mapsto \mathcal{A}_1(\tilde{\Gamma})\tilde{w} \end{aligned}$$

is a 24-fold covering map.

Proof. The only thing we have to prove is that $\mathcal{F}$ is the image of $SO(3) \times \mathcal{T}$ through Π. However, this is obvious, since

$$\Pi(SO(3)\times\mathcal{T}) = \mathcal{A}_1(SO(3))(\mathcal{T}) = \mathcal{A}_1(SO(3))\mathcal{A}_3(SO(3))(\mathcal{S}\cap\mathcal{C}_{\neq}) = \mathcal{F} \ . \tag{3.6.19}$$

□

End of proof of Theorem 3.6.3. The last step consists in factoring out the group action $\mathcal{A}_3$. This is easy, since the action $\mathcal{A}_3$ is free. Therefore, one can decompose

$$\mathcal{T} \ni \tilde{w} = \mathcal{A}_3(\tilde{R})w \quad (w \in \mathcal{S} \cap \mathcal{C}_{\neq}) \tag{3.6.20}$$

in a unique way, and moreover the map

$$\tilde{w} \mapsto (\tilde{R}, w)$$

is smooth. Therefore, a 24-fold covering of $\mathcal{F}$ is naturally induced by the map

$$(\tilde{\Gamma}, \tilde{R}, w) \mapsto \mathcal{A}_1(\tilde{\Gamma})\mathcal{A}_3(\tilde{R})w \ . \tag{3.6.21}$$

Now, setting

$$\Gamma := \tilde{\Gamma}\tilde{R}$$
$$R := \tilde{R}^{-1} \ ,$$

we find that also

$$(\Gamma, R, w) \mapsto \mathcal{A}_1(\Gamma)\mathcal{A}_2(R)w \tag{3.6.22}$$

is a 24-fold covering of $\mathcal{F}$, which completes the proof of (ii) and of Theorem 3.6.3.

3.6.2 Elastic potential energy

Let us study the form of the elastic potential energy and of the potential energy of self-gravitation in the coordinates just introduced. As for the triaxial case, with an abuse of terminology, we will call the sum of these two potential energies simply "elastic potential energy" and we will denote it by V_e; furthermore, we will refer to the corresponding forces as to the "elastic forces", leaving understood that they include also the forces related to self-gravitation.

In the equilibrium state, because of the rotational invariance, all three principal moments of inertia are equal to the same constant I_0. For simplicity, we use the differences between the I_j's and I_0 as configuration variables instead of the I_j's themselves, so we define

$$J_i := I_i - I_0 \ , \qquad\qquad (i = 1, 2, 3) \ , \tag{3.6.23}$$

and we assume (without loss of generality) that

$$z_j(v_0) = 0 \quad \forall j \ .$$

Remark 3.6.10. *Due to the $\mathcal{A}_1$- and $\mathcal{A}_2$-invariance, the elastic potential energy does not depend on Γ and R.*

As we did for the triaxial satellite, we also assume that the minimum is nondegenerate and that the body has very large moduli of elasticity.

Assumption 14. *The elastic potential energy has the form*

$$V_e(J, z) = \frac{1}{\epsilon} V_0(J, z) \equiv \frac{1}{\epsilon} \left[Q(J, z) + V_3(J, z) \right] \ , \tag{3.6.24}$$

where ϵ is a small parameter, Q a nondegenerate quadratic form and V_3 has a zero of order three at the origin.

We want to study more in detail the form of the elastic potential near the equilibrium, but we have to cope with the fact that our coordinates are singular at the equilibrium configuration $v_0(\mathbf{x}) = \mathbf{x}$.

Lemma 3.6.11. *The quadratic part of the elastic potential energy has the form*

$$\begin{aligned} Q(J,z) &= \frac{A}{2}({J_1}^2 + {J_2}^2 + {J_3}^2) + B(J_1J_2 + J_1J_3 + J_2J_3) + \\ &+ \sum_{j=1}^{+\infty} C_j z_j (J_1 + J_2 + J_3) + \frac{1}{2} \sum_{j,k=1}^{+\infty} D_{jk} z_j z_k \ , \quad (3.6.25) \end{aligned}$$

where the constants A, B, C_j, D_{jk} are such that the quadratic part $Q(J,z)$ is a positive definite quadratic form in the variables (J,z). In particular, this implies $A > B$.

Remark 3.6.12. *By Remark 3.6.1, such an expression can be used to compute the Lagrange equations only outside $\mathcal{C}_=$.*

Proof. Any $v \in \mathcal{F}$ can be represented as $\mathcal{A}_1(\Gamma)\mathcal{A}_2(R)w$, for some $\Gamma, R \in SO(3)$ and $w \in \mathcal{S} \cap \mathcal{C}_{\neq}$. Moreover, due to the group action invariance, the potential energy associated to the configuration v must be the same as the potential energy associated to the configuration w. Therefore the functional form of the elastic potential in terms of the variables J, z can be computed working in $\mathcal{S}$ and then the obtained form holds on the whole of $\mathcal{F}$.

The independence of the Lagrangian of the body with respect to the choice of the representative (Γ, R, w) from the covering mentioned in Proposition 3.6.3 implies that the expression of the elastic potential energy must be symmetric with respect to permutations of the indices $i = 1, 2, 3$, and the expression of $Q(J, z)$ in (3.6.25) is the most general expression of a quadratic form with such a property. □

3.6.3 Planar restriction

We will study the equations of motion under our usual assumptions of planar restriction (Assumptions 6, 7 and 8), but it is worth making some further remark that holds true for the spherically symmetric case only.

Remark 3.6.13. *As a consequence of Assumptions 7 and 8, R is a rotation about the $\boldsymbol{e}_3$-axis, i.e. there exists an angle β such that*

$$R = R(\beta) := \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.6.26}$$

Remark 3.6.14. *The assumptions 7 and 8, together with Theorem 3.6.3, imply that $(\alpha, \beta, I_1, I_2, I_3, z_1, z_2, \ldots)$ are good Lagrangian coordinates for the space of body configurations. Actually, by following the proof of Theorem 3.6.3 one can show that such coordinates form a 4-fold covering of the configuration space restricted to planar configurations.*

The fact that dynamics remains confined for all times within the set $\mathcal{F}$ will be guaranteed by the local stability result proved in the following sections.

3.6.4 Kinetic energy

Again, the kinetic energy T can be written as the sum of the kinetic energy of the center of mass

$$T_{cm} = \frac{m}{2}(\dot{R}^2 + R^2\dot{\psi}^2) \tag{3.6.27}$$

and the kinetic energy of the satellite with respect to its center of mass

$$T_r := T_r(\dot{\alpha}, \dot{\beta}, \dot{J}, \dot{z}; J, z) \ . \tag{3.6.28}$$

Remark 3.6.15. *T_r is independent of α and β due to the rotational ($\mathcal{A}_1$- and $\mathcal{A}_2$-) invariance of the satellite.*

We will use the notation

$$T_r := \frac{1}{2}\sum_{i,k=1}^{+\infty} a_{ik}(J, z)\dot{q}_i\dot{q}_k \ , \tag{3.6.29}$$

where $q = (\alpha, \beta, J, z)$. Observe that the coefficients $a_{ik}(J, z)$ are such that the quadratic form is positive definite on $\mathcal{F}$.

Lemma 3.6.16. *The coefficient $a_{11}(J, z)$ satisfies*

$$a_{11}(J, z) = I_3 = I_0 + J_3 \ . \tag{3.6.30}$$

Proof. For $w \in \mathcal{S}$, we set

$$u = \mathcal{A}_2(R(\beta))w \; . \tag{3.6.31}$$

Now, let us evaluate the kinetic energy T_r. For $v \in \mathcal{F}$ we have

$$v(\mathbf{x}) = \Gamma(\alpha)u(\mathbf{x}) \; . \tag{3.6.32}$$

Taking the derivative with respect to time, we get

$$\dot{v}(\mathbf{x}) = \frac{d\Gamma(\alpha)}{dt} + u(\mathbf{x}) + \Gamma(\alpha)\dot{u}(\mathbf{x}) \; . \tag{3.6.33}$$

Therefore,

$$\begin{aligned} T_r &= \frac{1}{2}\int_\Omega [\dot{v}(\mathbf{x})]^2 \rho(\mathbf{x}) d^3\mathbf{x} = \frac{1}{2}\int_\Omega [\Gamma(-\alpha)\dot{v}(\mathbf{x})]^2 \rho(\mathbf{x}) d^3\mathbf{x} = \\ &= \frac{1}{2}\int_\Omega \left[\Gamma(-\alpha)\frac{d\Gamma(\alpha)}{dt}u(\mathbf{x}) + \dot{u}(\mathbf{x})\right]^2 \rho(\mathbf{x}) d^3\mathbf{x} = \\ &= \frac{1}{2}\int_\Omega [\omega \times u(\mathbf{x})]^2 \rho(\mathbf{x}) d^3\mathbf{x} + \qquad (3.6.34) \\ &\quad + \int_\Omega \langle \omega \times u(\mathbf{x}), \dot{u}(\mathbf{x})\rangle \rho(\mathbf{x}) d^3\mathbf{x} + \frac{1}{2}\int_\Omega [\dot{u}(\mathbf{x})]^2 \rho(\mathbf{x}) d^3\mathbf{x}, \end{aligned}$$

where ω is the angular velocity of the satellite.

Then the thesis follows as in Lemma 3.5.6. □

3.6.5 Lagrangian and reduction

Now we can write the explicit form of the Lagrangian function.

Our Lagrangian coordinates are:

- the polar coordinates (R, ψ) of the center of mass of the satellite;
- the angle $\chi = \alpha - \psi$, describing the rigid rotation of the satellite, measured with respect to the line of centers;
- the angle β, associated to the action $\mathcal{A}_2$, which describes how the principal axes of inertia rotate, with respect to the rigid motion of the satellite;
- the coordinates (J,z), which parameterize the configuration $w \in \mathcal{F}$.

The Lagrangian, again, has the structure

$$\mathcal{L} = T_{cm} + T_r - V_g - V_e \ . \tag{3.6.35}$$

After noticing that now the angle γ of equation (3.4.1) equals $\chi + \beta$, the Lagrangian function assumes therefore the following form:

$$\begin{aligned} \mathcal{L} &= \frac{m}{2}\left(\dot{R}^2 + R^2\dot{\psi}^2\right) + T_r(\dot{\chi} + \dot{\psi}, \dot{J}, \dot{\beta}, \dot{z}; J, z) + \frac{GMm}{R} + \\ &+ \frac{GM}{R^3}\left[J_1 - 2J_2 + J_3 + 3(J_2 - J_1)\cos^2(\chi + \beta)\right] - V_e(J, z; \epsilon) \ . \end{aligned}$$

Again, the variable ψ is cyclic and we pass to the reduced Lagrangian $\mathcal{L}^*$. We have the conservation of the total angular momentum

$$p := \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = mR^2\dot{\psi} + (J_3 + I_0)(\dot{\chi} + \dot{\psi}) + 2\sum_{k=2}^{5+n} a_{1k}(J, z)\dot{q}_k \ , \tag{3.6.36}$$

and then we find

$$\dot{\psi} = \frac{p - (J_3 + I_0)\dot{\chi} - 2\sum_{k=2}^{5+n} a_{1k}(J,z)\dot{q}_k}{mR^2 + I_0 + J_3} \ . \tag{3.6.37}$$

Hence,

$$\mathcal{L}^* = \mathcal{L} - \dot{\psi}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} = T_2 + T_1 - \tilde{V} \ , \tag{3.6.38}$$

where

$$\begin{aligned}
T_2 &= \frac{m}{2}\dot{R}^2 + T_r(\dot{\chi}, \dot{J}, \dot{\beta}, \dot{z}; J, z) + \\
&\quad - \frac{\left[(J_3 + I_0)\dot{\chi} + 2\sum_{k=2}^{5+n} a_{1k}(J,z)\dot{q}_k\right]^2}{2(mR^2 + I_0 + J_3)} \\
T_1 &= \frac{p\left[(J_3 + I_0)\dot{\chi} + 2\sum_{k=2}^{5+n} a_{1k}(J,z)\dot{q}_k\right]}{mR^2 + I_0 + J_3} \\
\tilde{V} &= \frac{p^2}{2(mR^2 + I_0 + J_3)} - \frac{GMm}{R} + \\
&\quad - \frac{GM}{R^3}\left[J_1 - 2J_2 + J_3 + 3(J_2 - J_1)\cos^2\gamma\right] + V_e(J, z; \varepsilon) \ .
\end{aligned}$$

Again, we will exploit the conservation of the energy

$$E := T_2 + \tilde{V} = \sum_{k=1}^{5+n} \dot{y}_k \frac{\partial \mathcal{L}^*}{\partial \dot{y}_k} - \mathcal{L}^* \ , \tag{3.6.39}$$

where

$$y := (R, \chi, \beta, J, z)$$

and the fact that the strict minima of $\tilde{V}$ are Lyapunov-stable equilibria of the reduced system.

Let R_0 be a nondegenerate minimum of the function

$$V_{G0}(R) := -\frac{GMm}{R} + \frac{p^2}{2(mR^2 + I_0)} \ . \tag{3.6.40}$$

Then we have the following

Lemma 3.6.17. *For any ϵ small enough there exist $\bar{R}, \bar{J}, \bar{z}$, s.t.*

(1) the manifold

$$\mathcal{M} := \left\{ (\bar{R}, \chi, \beta, \bar{J}_1, \bar{J}_2, \bar{J}_3, \bar{z}) | \chi + \beta = 0 \right\} \ ,$$

is composed by critical points of $\tilde{V}$.

(2) $\mathcal{M}$ is a minimum of $\tilde{V}$ which is nondegenerate in the transversal direction.

(3) One has $(\bar{J}, \bar{z}) = O(\epsilon)$ and $|\bar{R} - R_0| = O(\epsilon)$.

(4) Finally $\bar{J}_1 < \bar{J}_2 < \bar{J}_3$.

Remark 3.6.18. *Point (4) guarantees that $\mathcal{M} \subset \mathcal{C}_{\neq}$. If ϵ is sufficiently small, then we have $\mathcal{M} \subset \mathcal{F}$.*

Remark 3.6.19. *$\mathcal{M}$ is the manifold corresponding to 1:1 spin orbit resonance.*

Proof. We look for a minimum of $\tilde{V}$ in the domain $J_1 \leq J_2$ and $|J| \leq C\epsilon$ for some fixed C.

First remark that, as a function of $\gamma = \chi + \beta$, $\tilde{V}$ has a minimum at γ=0 (strict if $J_1 < J_2$). Consider now $\tilde{V}\Big|_{\gamma=0}$; as a function of R it has a nondegenerate minimum at some point $R = R(J, z)$ fulfilling

$$|R(J, z) - R_0| \leq C\epsilon \ .$$

Consider now the restriction $\bar{V} = \bar{V}(J, z)$ of $\tilde{V}$ to the manifold $\gamma = 0$, $R = R(J, z)$; since

$$\bar{V}(J, z) = \frac{1}{\epsilon}\left[Q(J, z) + V_3(J, z)\right] + O(1) \ , \tag{3.6.41}$$

such a function has a nondegenerate minimum close to zero.

Then (1), (2) and (3) follow provided one shows that $\bar{J}_1 < \bar{J}_2$. We are now going to prove (4) which in particular implies the thesis.

To this end, observe that at the critical point one has

$$0 = \frac{\partial \tilde{V}}{\partial J_1} = \frac{2GM}{\bar{R}^3} + \frac{A}{\epsilon}\bar{J}_1 + \frac{B}{\epsilon}(\bar{J}_2 + \bar{J}_3) + \frac{1}{\epsilon}\sum_{j=1}^{+\infty} C_j \bar{z}_j + O(\epsilon) \tag{3.6.42}$$

$$0 = \frac{\partial \tilde{V}}{\partial J_2} = -\frac{GM}{\bar{R}^3} + \frac{A}{\epsilon}\bar{J}_2 + \frac{B}{\epsilon}(\bar{J}_1 + \bar{J}_3) + \frac{1}{\epsilon}\sum_{j=1}^{+\infty} C_j \bar{z}_j + O(\epsilon) \ . \tag{3.6.43}$$

$$\begin{aligned} 0 \quad = \quad & \frac{\partial \tilde{V}}{\partial J_3} = -\frac{p^2}{2(m\bar{R}^2 + I_0 + \bar{J}_3)^2} - \frac{GM}{\bar{R}^3} + \\ & + \frac{A}{\epsilon}\bar{J}_3 + \frac{B}{\epsilon}(\bar{J}_1 + \bar{J}_2) + \frac{1}{\epsilon}\sum_{j=1}^{+\infty} C_j \bar{z}_j + O(\epsilon) \ . \end{aligned} \tag{3.6.44}$$

Subtracting (3.6.43) from (3.6.42), we obtain

$$\frac{3GM}{\bar{R}^3} + \frac{A-B}{\epsilon}(\bar{J}_1 - \bar{J}_2) + O(\epsilon) = 0 \ . \tag{3.6.45}$$

The positive definiteness of the quadratic form Q implies $A - B > 0$. Therefore, if ϵ is sufficiently small, we have $\bar{J}_1 < \bar{J}_2$. Subtracting (3.6.44) from (3.6.43), we get

$$\frac{p^2}{2(m\bar{R}^2 + I_0 + \bar{J}_3)^2} + \frac{A-B}{\epsilon}(\bar{J}_2 - \bar{J}_3) + O(\epsilon) = 0 \ . \tag{3.6.46}$$

Hence, if ϵ is sufficiently small, we have $\bar{J}_2 < \bar{J}_3$. □

Corollary 3.6.20. *The critical submanifold of the phase space manifold*

$$\mathfrak{M} := \{(y, \dot{y}) | y \in \mathcal{M}, \dot{y} = 0\}$$

is stable in the sense of Definition 2.3.4 for the Lagrangian system of equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}^*}{\partial \dot{y}_k} = \frac{\partial \mathcal{L}^*}{\partial y_k} \ . \qquad (k = 1, 2, \ldots) \tag{3.6.47}$$

Corollary 3.6.21. *The two-dimensional submanifold of the phase space of the unreduced system, obtained as a one-parameter family of synchronous resonant orbits,*

$$\mathfrak{O} := \left\{(y, \psi, \dot{y}, \dot{\psi}) | (y, \dot{y}) \in \mathfrak{M}, \psi \in S^1, \dot{\psi} = \bar{\dot{\psi}}\right\} \ ,$$

with $\bar{\dot{\psi}}$ *given by plugging* $(y, \dot{y}) \in \mathfrak{M}$ *into* (3.6.37) *(notice that this substitution gives a well-defined result, since the r.h.s. of* (3.6.37) *is* *independent of both the configuration variables χ and β), is stable in the sense of Definition 2.3.4 for the unreduced Lagrangian system of equations*

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_k} = \frac{\partial \mathcal{L}}{\partial x_k} \ . \qquad (k = 1, 2, \ldots) \tag{3.6.48}$$

3.6.6 Dissipative dynamics

As we did for the triaxial case, we now modify the Euler-Lagrange equations in order to take account of internal friction. We study the system of equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_k} - \frac{\partial \mathcal{L}}{\partial x_k} = -f_k(\dot{x}, x) \ , \qquad (k = 1, 2, \ldots) \tag{3.6.49}$$

Again, we observe that $f_2(\dot{x}, x)$ must be identically zero and that all the f_k's must be independent of ψ and $\dot{\psi}$. Then we can study the reduced system

$$\frac{d}{dt}\frac{\partial \mathcal{L}^*}{\partial \dot{y}_k} - \frac{\partial \mathcal{L}^*}{\partial y_k} = -\tilde{f}_k(\dot{y}, y) \ . \qquad (k = 1, 2, \ldots) \tag{3.6.50}$$

Then, we assume that the $\tilde{f}_k$'s represent a dissipation.

Assumption 15. *The functional form of the functions $\tilde{f}_k$ is such that*

$$\sum_{k=1}^{+\infty} \dot{y}_k \tilde{f}_k(\dot{y}, y) \geq 0 \ . \tag{3.6.51}$$

.

Then, the usual reasoning about non-increasing energy, together with Lemma 3.5.7, gives the result of orbital stability of the synchronous resonance, in the following sense.

Theorem 3.6.22. *The manifold $\mathfrak{M}$ is stable in the future, in the sense of Definition 2.3.4, for the reduced system of equations* (3.6.50)*.*

Corollary 3.6.23. *The manifold $\mathfrak{O}$ of synchronous resonant circular orbits is orbitally stable in the future, in the sense of Definition 2.3.4, for the unreduced system of equations* (3.6.49)*.*

3.6.7 Multi-layer satellite

We remark that the proof of the orbital stability that we have given for a spherically symmetric satellite can be immediately extended to the case of a *multi-layer* spherically symmetric satellite. A spherically symmetric satellite with n layers has the following structure in the reference configuration: the inner layer (the core) is a sphere, while the $n-1$ outer layers are spherical shells, adjacent to one another. These n layers are free to slide (possibly with some dissipation of energy) on one another. Moreover, we require the same properties of invariance of the Lagrangian function as for the "single-layer" case.

All the machinery works exactly the same way, except some technical details. In this case, it is convenient to use the group actions $\mathcal{A}_1$ and $\mathcal{A}_3$, instead of $\mathcal{A}_1$ and $\mathcal{A}_2$. Then, observe that $\mathcal{A}_1[R(\alpha)]\mathcal{A}_2[R(\beta)] = \mathcal{A}_1[R(\alpha+\beta)]\mathcal{A}_3[R(-\beta)]$. Then, when passing to the quotient, one introduces angles $\theta := \alpha + \beta$ and $\phi := -\beta$. In order to handle the n layers, one has to define n group actions $\mathcal{A}_{31}, \dots, \mathcal{A}_{3n}$, instead of the single action $\mathcal{A}_3$, each of these n group actions corresponding to the rotational symmetry of only one of the layers. In particular, the action $\mathcal{A}_{3k}$ will correspond to the k-th layer of the satellite sliding between the two adjacent layers, while all the other layers do not move. Then, instead of passing to the quotient with respect to the action $\mathcal{A}_3$ and describing the group action through an angle ϕ, one passes to the quotient with respect to all the group actions $\mathcal{A}_{3k}$ and correspondingly defines n angles $\phi_1, \dots, \phi_n$. Then, in order to refer rotations to the line of centers, thus making the Lagrangian of the system independent from the cyclic coordinate ψ, one defines $\tilde{\theta} := \theta - \psi$.

Now, we observe that, in terms of the variables $(\tilde{\theta}, \phi)$, the manifold $\mathcal{M}$ of equilibria of the reduced system for the single-layer satellite corresponds to $\{R = \bar{R}, \tilde{\theta} = 0, \phi \in S^1, J = \bar{J}, z = \bar{z}\}$. In the case of the multi-layer satellite, one simply finds that the manifold of equilibria is the n-dimensional manifold $\{R = \bar{R}, \tilde{\theta} = 0, \phi \in \mathbb{T}^n, J = \bar{J}, z = \bar{z}\}$, where $\phi := (\phi_1, \dots, \phi_n)$ and $\mathbb{T}^n$ is the n-dimensional torus.

With these slight changes, all the stability results that we have obtained hold also for the case of a multi-layer satellite.

3.7 General comments

We recall the theorem of the three outcomes that we have proved in the previous chapter: one of the consequences was that, if there is orbital stability of the synchronous resonance, then the synchronous resonance is also asymptotically stable. The results of orbital stability that we have proved in the present chapter are not a completely rigorous proof of the asymptotic stability, since in the present chapter we have done some approximations. If the same result of the orbital stability that we have obtained in this chapter were proved removing our approximations, then the asymptotic stability of the synchronous resonance would be automatically proved through the theorem of the three outcomes.

A further comment is worth being done about the result of stability that we have obtained for a spherically symmetric satellite: we are not able to prove that each of the synchronous resonant orbits of the manifold $\mathfrak{O}$ is orbitally stable, when considered alone. Even if we have proved that the speed of rotation of the satellite tends to zero, we are unable to prove that asymptotically "the satellite stops rotating": we only prove that asymptotically the principal axes of inertia stop rotating. This is compatible with a situation in which the satellite has a constant shape, but the direction of the deformation continuously changes in the body. In a pictorial way one can think of a rubber balloon which on a wooden cross whose axes are fixed in space. In a more rigorous fashion, we are stating that the previous theorem implies that $\chi + \beta \to 0$ when $t \to +\infty$, but it says nothing about the individual behavior of χ and β. If there is asymptotic stability, then this implies that $\dot{\chi}$ and $\dot{\beta}$ approach zero as $t \to +\infty$ (the rubber balloon slides on the wooden cross more and more slowly). This, however, does not imply that there exist $\bar{\chi}$ and $\bar{\beta}$ such that $\chi \to \bar{\chi}$ or that $\beta \to \bar{\beta}$, since, at our level of generality, we are not able to prove the convergence of the integrals

$$\int_0^{+\infty} \dot{\chi}(t)dt \qquad \int_0^{+\infty} \dot{\beta}(t)dt \ .$$

3.8 A direct proof of the asymptotic stability

In this section, we show that, under some more assumptions and approximations, in the *spherically symmetric case* it is possible to give a proof of the asymptotic stability, directly from the equations of motion.

The further approximation that we impose here is a finite-dimensionality approximation.

Assumption 16 (Finite-dimensionality)**.** *The space $\mathcal{C}$ is finite-dimensional.*

We are assuming that the dimension of $\mathcal{C}$ is arbitrary, but finite. For example, one can obtain this by cutting at any order the multipole expansion of the configuration $v \in \mathcal{C}$. The Lagrangian variables $(z_1, z_2, \ldots)$ will therefore be replaced by $(z_1, z_2, \ldots, z_n)$.

Under this finite-dimensionality assumption, we can prove a useful result about the expression of the kinetic energy.

Lemma 3.8.1. *The coefficient $a_{12}(J,z)$ satisfies:*

$$\lim_{(J,z)\to 0} a_{12}(J,z) = 0 \ . \tag{3.8.1}$$

Proof. We use the same notation as in Lemma 3.6.16. Then, reasoning as in the proof of Lemma 3.6.16, we observe that the coefficient a_{12} of the term $\dot\alpha\dot\beta$ in the expression of the kinetic energy arises from the integral

$$\int_\Omega \langle \omega \times u(\mathbf{x}), \dot u(\mathbf{x})\rangle \rho(\mathbf{x}) d^3\mathbf{x}$$

in (3.6.34). The $\dot\alpha$ factor comes from the angular velocity ω, while the $\dot\beta$ factor is hidden in $\dot u(\mathbf{x})$. Define

$$w_{def} = w - v_0 = \mathcal{A}_2[R(-\beta)]u - v_0 \ . \tag{3.8.2}$$

We get

$$\begin{aligned} \dot u(\mathbf{x}) \quad &= \quad R(\beta)\left\{\dot w_{def}\left[R(-\beta)\mathbf{x}\right]\right\} + \dot\beta \frac{\partial R(\beta)}{\partial \beta} w_{def}\left[R(-\beta)\mathbf{x}\right] + \\ & \quad + \dot\beta R(\beta)\nabla w_{def}\left[R(-\beta)\mathbf{x}\right] \cdot \frac{\partial R(-\beta)}{\partial\beta}\mathbf{x} \ . \end{aligned} \tag{3.8.3}$$

Here, we notice that the first of the three addenda is independent of $\dot\beta$ (since $w = w_{def} + v_0$ belongs to the section $\mathcal{S}$, which is transversal to the group action $\mathcal{A}_2$), so we have (exploiting $v_0(\mathbf{x}) = \mathbf{x}$)

$$a_{12}(J,z) = \frac{1}{2}\int_\Omega \rho(\mathbf{x})\langle \mathbf{u}_3 \times \{\mathbf{x} + R(\beta) w_{def}[R(-\beta)\mathbf{x}]\}, \tag{3.8.4}$$

$$\frac{\partial R(\beta)}{\partial\beta} w_{def}\left[R(-\beta)\mathbf{x}\right] + R(\beta)\nabla w_{def}\left[R(-\beta)\mathbf{x}\right] \cdot \frac{\partial R(-\beta)}{\partial\beta}\mathbf{x}\rangle d^3\mathbf{x} \ ,$$

which goes to zero when $w_{def} \to 0$, i.e. when $(J, z) \to 0$; this is where we exploit the assumption of finite-dimensionality, since we use the equivalence of norms ($w_{def} \to 0 \Leftrightarrow \nabla w_{def} \to 0$). $\square$

Moreover, we will make some more assumption about the dissipative terms in the equations of motion, i.e. we introduce a Rayleigh's dissipation function $F(\dot{y}; y)$. Namely, we assume that the equations of motion of the reduced system have the form (3.8.5) below. Of course, we do not expect dissipation to act directly on the orbital variables R and χ, so we assume F to be independent of R, χ, $\dot{R}$ and $\dot{\chi}$. Due to the $\mathcal{A}_2$-invariance of the satellite, F is also independent of the coordinate β (but it depends on $\dot{\beta}$); however, for the proof of our result, it is not necessary that F be β-independent.

Assumption 17. *As a function of the velocities $(\dot{\beta}, \dot{J}, \dot{z})$, the Rayleigh's dissipation function $F(\dot{\beta}, \dot{J}, \dot{z}; J, z)$ has a nondegenerate minimum at 0.*

We denote by $y^e := (\beta, J, z)$ the variables fixing the configuration of the satellite. So in particular we have $F = F(y^e, \dot{y}^e)$ (with $\partial F / \partial \beta = 0$).

Then the result of asymptotic stability is:

Theorem 3.8.2. *If ϵ is sufficiently small, the manifold $\mathfrak{M}$, defined in Section 3.6, is asymptotically stable for the dynamical system of equations*

$$\frac{d}{dt}\frac{\partial \mathcal{L}^*}{\partial \dot{y}_k} - \frac{\partial \mathcal{L}^*}{\partial y_k} = -\frac{\partial F}{\partial \dot{y}_k}\ , \qquad (k = 1, 2, \ldots) \qquad (3.8.5)$$

with $\mathcal{L}^*$ defined as in Section 3.6.

Proof. Let us evaluate the time derivative of the energy:

$$\begin{aligned}
\frac{dE}{dt} &= \frac{d}{dt}\left(\sum_{k=1}^{4+n} \dot{y}_k \frac{\partial \mathcal{L}^*}{\partial \dot{y}_k}\right) - \frac{d}{dt}\mathcal{L}^* = \\
&= \sum_{k=1}^{4+n} \ddot{y}_k \frac{\partial \mathcal{L}^*}{\partial \dot{y}_k} + \sum_{k=1}^{4+n} \dot{y}_k \frac{d}{dt}\frac{\partial \mathcal{L}^*}{\partial \dot{y}_k} - \sum_{k=1}^{4+n} \dot{y}_k \frac{\partial \mathcal{L}^*}{\partial y_k} - \sum_{k=1}^{4+n} \ddot{y}_k \frac{\partial \mathcal{L}^*}{\partial \dot{y}_k} = \\
&= -\sum_{k=1}^{4+n} \dot{y}_k \frac{\partial F}{\partial \dot{y}_k} = -\left(2Q_F + \sum_{k=1}^{4+n} \dot{y}_k \frac{\partial F_3}{\partial \dot{y}_k}\right) ,
\end{aligned} \tag{3.8.6}$$

where Q_F is the quadratic part of F, and F_3 is the part of order 3 in $\dot{y}^e$. Let $\mathcal{ND} := \{(y, \dot{y}) | F(\dot{y}; y) = 0\} = \left\{(y, \dot{y}) | \dot{\beta} = \dot{J} = \dot{z} = 0\right\}$ be the subset of phase space where there is no energy dissipation. Then, due to LaSalle's invariance principle, any solution such that $(y(0), \dot{y}(0))$ belongs to a sufficiently small neighborhood of $\mathfrak{M}$ (notice that such a solution will stay bounded for all $t \geq 0$ due to the stability of $\mathcal{M}$ proved in Section 3.6) will get arbitrarily close to the largest invariant subset of $\mathcal{ND}$, for $t \to +\infty$. Therefore, the only thing we have to check is that the set $\mathcal{ND}$ contains no orbit, apart from the points of the manifold $\mathcal{M}$. To check this, observe that, if such an orbit existed, it would satisfy equations (3.8.5). In particular, the orbit satisfies

$$\frac{d}{dt}\frac{\partial \mathcal{L}^*}{\partial \dot{\chi}} - \frac{\partial \mathcal{L}^*}{\partial \chi} = 0 \tag{3.8.7}$$

and

$$\frac{d}{dt}\frac{\partial \mathcal{L}^*}{\partial \dot{\beta}} - \frac{\partial \mathcal{L}^*}{\partial \beta} = -\frac{\partial F}{\partial \dot{\beta}} \ . \tag{3.8.8}$$

When restricting to $\mathcal{ND}$, these two equations become, respectively,

$$\begin{aligned} -\frac{(I_0+J_3)^2\ddot{\chi}}{mR^2+I_0+J_3}+\frac{2mR(I_0+J_3)^2\dot{\chi}\dot{R}}{(mR^2+I_0+J_3)^2}+ & & \\ +(I_0+J_3)\ddot{\chi}-\frac{2pmR(I_0+J_3)\dot{R}}{(mR^2+I_0+J_3)^2} &= & -\frac{\partial\tilde{V}}{\partial\chi}(y) \end{aligned} \tag{3.8.9}$$

and

$$\begin{aligned} -\frac{a_{12}(J,z)(I_0+J_3)\ddot{\chi}}{mR^2+I_0+J_3}+\frac{2mRa_{12}(J,z)(I_0+J_3)\dot{\chi}\dot{R}}{(mR^2+I_0+J_3)^2}+ & & \\ +a_{12}(J,z)\ddot{\chi}-\frac{2pmRa_{12}(J,z)\dot{R}}{(mR^2+I_0+J_3)^2} &= & -\frac{\partial\tilde{V}}{\partial\beta}(y)\ , \end{aligned} \tag{3.8.10}$$

where we took into account that, by Assumption 17, $\partial F/\partial\dot{\beta}$ vanishes on $\mathcal{M}$. Observe that the r.h.s.'s of the two equations are equal.

Multiplying (3.8.9) by $\frac{a_{12}(J,z)}{I_0+J_3}$ and subtracting (3.8.10) we get

$$\left(1-\frac{a_{12}}{I_0+J_3}\right)\frac{\partial\tilde{V}}{\partial\gamma}=0\ ,$$

which, by (3.8.1), implies $\frac{\partial\tilde{V}}{\partial\gamma}=0$, and therefore $\chi+\beta=0$. Then, we have $\dot{\chi}=-\dot{\beta}=0$, since $\dot{\beta}=0$ on $\mathcal{ND}$. Now , substituting $\dot{\chi}=\ddot{\chi}=0$ into equations (3.8.9) and (3.8.10), we find $\dot{R}=0$. Finally, we observe that now we have

$$\dot{\chi}=\dot{\beta}=\dot{R}=\dot{J}=\dot{z}=0\ , \tag{3.8.11}$$

which is true on the equilibrium manifold $\mathcal{M}$ only.

We have thus proved that the only orbits contained in $\mathcal{ND}$ are the points of the synchronous resonance manifold $\mathcal{M}$, which, by LaSalle's invariance principle, implies the asymptotic stability of $\mathcal{M}$. □

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